$\boxed{2.3}$

$$A = (a_{ij})_{m \times n} = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

$\underbrace{\hphantom{(a_{ij})_{m \times n}}}_{}$ short-hand for

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ & & \vdots & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} = \begin{array}{c c} & j \\ i & \begin{bmatrix} & a_{1j} & \\ & a_{2j} & \\ & \vdots & \\ a_{i1} \ a_{i2} \cdots & a_{ij} & \cdots \ a_{in} \\ & \vdots & \\ & a_{mj} & \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} & | & \\ — & a_{ij} & — \\ & | & \end{bmatrix}_{m \times n}$$

<u>def</u> $A + B$ both same size $m \times n$

$A = (a_{ij})_{m \times n}$ $\quad B = (b_{ij})_{m \times n}$

define $A + B = (c_{ij})_{m \times n}$ where $\boxed{c_{ij} = a_{ij} + b_{ij}}$

for $1 \le i \le m$
for $1 \le j \le n$

Shorten: $A + B = (a_{ij} + b_{ij})_{m \times n}$

$$A + B = \begin{bmatrix} & \vert & \\ — & a_{ij} & — \\ & \vert & \end{bmatrix}_{m\times n} + \begin{bmatrix} & \vert & \\ — & b_{ij} & — \\ & \vert & \end{bmatrix}_{m\times n}$$

$$= \begin{bmatrix} & \vert & \\ — & a_{ij}+b_{ij} & — \\ & \vert & \end{bmatrix}_{m\times n}$$

Similarly, • $A - B = (a_{ij} - b_{ij})_{m\times n}$ ✓

• $kA = k(a_{ij})_{m\times n} = (k a_{ij})_{m\times n} = \begin{pmatrix} & \vert & \\ — & k a_{ij} & — \\ & \vert & \end{pmatrix}$

<u>Example</u> $T_1 : \mathbb{R}^3 \to \mathbb{R}^2$

$$T_1\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x - 2y + 5z \\ 2x + y - 3z \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & -2 & 5 \\ 2 & 1 & -3 \end{bmatrix}}_{2\times 3} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$T_2 : \mathbb{R}^3 \to \mathbb{R}^2$

$$T_2\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 4x + 7y - 2z \\ x - y + 4z \end{bmatrix} = \begin{bmatrix} 4 & 7 & -2 \\ 1 & -1 & 4 \end{bmatrix}_{2\times 3} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\boxed{[T_1 + T_2] = \begin{bmatrix} 7 & 5 & 3 \\ 3 & 0 & 1 \end{bmatrix}_{2\times 3}}$$

From def:

$$\left(T_1+T_2\right)\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T_1\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) + T_2\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) \quad (\text{def})$$

$$= \begin{bmatrix} 3x-2y+5z \\ 2x+y-3z \end{bmatrix} + \begin{bmatrix} 4x+7y-2z \\ x-y+4z \end{bmatrix}$$

$$= \langle 3x-2y+5z,\ 2x+y-3z \rangle$$

$$+ \langle 4x+7y-2z,\ x-y+4z \rangle$$

$$= \boxed{\langle 7x+5y+3z,\ 3x+0y+z \rangle}$$

$$= \begin{bmatrix} 7 & 5 & 3 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$[T_1+T_2]$$

Proof of $[T_1+T_2] = [T_1] + [T_2]$:

Assume $T_1, T_2 : \mathbb{R}^n \to \mathbb{R}^m$.

Let $[T_1] = A = (a_{ij})_{m\times n}$ & $[T_2] = B = (b_{ij})_{m\times n}$

WTS $\forall \vec{v} \in \mathbb{R}^n$: $(T_1+T_2)\vec{v} = A\vec{v} + B\vec{v}$

Let $\vec{v} \in \mathbb{R}^n$ write $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$

- $A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}$ so $\vec{a}_j$ cols of A

- $B = \begin{bmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \\ | & | & & | \end{bmatrix}$ so $\vec{b}_j$ cols of B

$$(T_1 + T_2)\vec{v} = T_1(\vec{v}) + T_2(\vec{v}) \quad \text{(by def of } T_1 + T_2\text{)}$$

$$= A\vec{v} + B\vec{v} \quad \text{(by EQUIV of LT \& Mat Thm)}$$

"row picture" $= \begin{bmatrix} & | & \\ \cdots & \vec{a}_j & \cdots \\ & | & \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} & | & \\ \cdots & \vec{b}_j & \cdots \\ & | & \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

"col picture" $= (v_1\vec{a}_1 + \cdots + v_n\vec{a}_n) + (v_1\vec{b}_1 + \cdots + v_n\vec{b}_n)$ (in $\mathbb{R}^m$)

$$= v_1(\vec{a}_1 + \vec{b}_1) + \cdots + v_n(\vec{a}_n + \vec{b}_n) \quad \text{(associative prop. + right dist prop of } \mathbb{R}^m\text{)}$$

"row pic" $= \begin{bmatrix} & | & \\ \cdots & \vec{a}_j + \vec{b}_j & \cdots \\ & | & \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

$$= (a_{ij} + b_{ij})\vec{v} = (A + B)\vec{v} \quad \text{(def } A+B\text{)}$$

so since $\vec{v}$ was arbitrary:

$$[T_1+T_2] = A+B = [T_1]+[T_2].$$

$\square$

Compositions of LTs

recall $f: \mathbb{R} \to \mathbb{R}$

$g: \mathbb{R} \to \mathbb{R}$

$$(f \circ g)(x) = f(g(x))$$

& $$(g \circ f)(x) = g(f(x))$$

$T_1: \mathbb{R}^n \longrightarrow \mathbb{R}^k$ $\qquad \vec{v} \in \mathbb{R}^n \longrightarrow T_1(\vec{v}) \in \mathbb{R}^k$

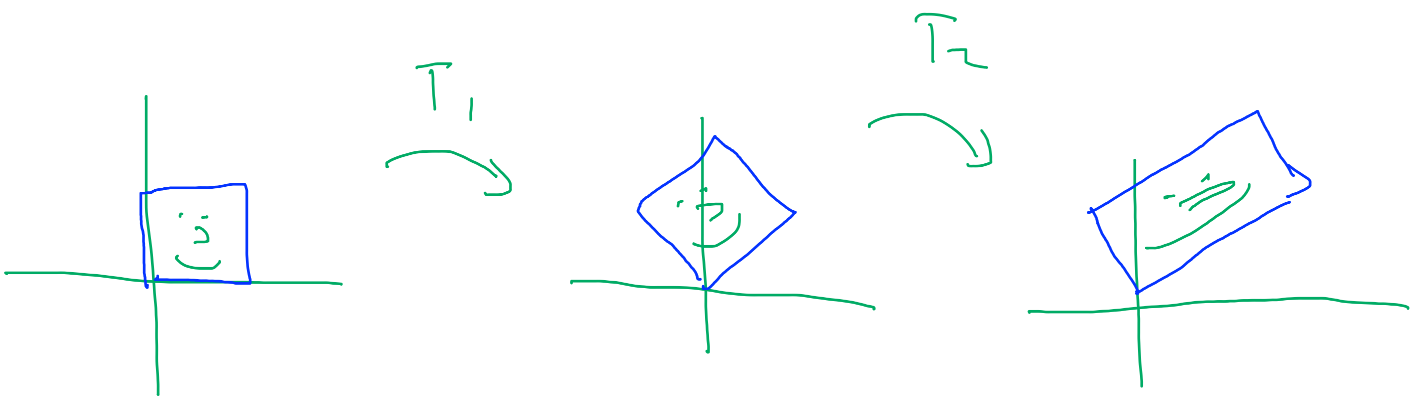
$T_2: \mathbb{R}^k \longrightarrow \mathbb{R}^m$

def $$(T_2 \circ T_1)(\vec{v}) = T_2(\underbrace{T_1(\vec{v})}_{\in \mathbb{R}^k}) \in \mathbb{R}^m$$

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "T_1"] \arrow[dr, dashed, "T_2 \circ T_1"'] & \boxed{\mathbb{R}^k} \arrow[d, "T_2"] \\
& \mathbb{R}^m
\end{tikzcd}$$

Warning

$\mathrm{Dom}(T_1) = \mathrm{Dom}(T_2)$



$T_1 = \mathrm{rot}_{\pi/4}$ $\qquad$ $T_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

Thm $T_2 \circ T_1$ is a LT.

$$T_2 \circ T_1 : \mathbb{R}^n \to \mathbb{R}^k \to \mathbb{R}^m$$

Pf • Add Prop: let $\vec{u}, \vec{v} \in \mathbb{R}^n$.

$$(T_2 \circ T_1)(\vec{u} + \vec{v}) = T_2\left(T_1(\vec{u} + \vec{v})\right) \quad \left(\text{def. of } T_2 \circ T_1\right)$$

$$= T_2\Big(\underbrace{\boxed{T_1(\vec{u})}}_{\in \mathbb{R}^k} + \underbrace{\boxed{T_1(\vec{v})}}_{\in \mathbb{R}^k}\Big) \quad (\text{b/c } T_1 \text{ is LT})$$

$$= T_2(\boxed{}) + T_2(\boxed{}) \quad (\text{b/c } T_2 \text{ is LT})$$

$$= T_2(T_1(\vec{u})) + T_2(T_1(\vec{v}))$$

$$= (T_2 \circ T_1)(\vec{u}) + (T_2 \circ T_1)(\vec{v}) \quad (\text{def of } T_2 \circ T_1)$$

- Scal product: exercise.

$\square$

Ex $T_1 : \mathbb{R}^3 \to \mathbb{R}^2$

$T_2 : \mathbb{R}^2 \to \mathbb{R}^4$

$$T_1\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - y + z \\ 3x + y - z \end{bmatrix} \qquad [T_1] = \underbrace{\begin{bmatrix} 1 & -1 & 1 \\ 3 & 1 & -1 \end{bmatrix}}_{2\times 3} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$T_2\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} u + 2v \\ 5u - v \\ 3u \\ u + v \end{bmatrix} \qquad [T_2] = \underbrace{\begin{bmatrix} 1 & 2 \\ 5 & -1 \\ 3 & 0 \\ 1 & 1 \end{bmatrix}}_{4\times 2} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$T_2 \circ T_1 : \mathbb{R}^3 \to \mathbb{R}^4$$

$$(T_2 \circ T_1)\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T_2\left(T_1\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)\right)$$

$$= T_2\left(\begin{bmatrix} \boxed{x - y + z} \\ \boxed{3x + y - z} \end{bmatrix}\right) = \begin{bmatrix} 7x + y - z \\ 2x - 6y + 6z \\ 3x - 3y + 3z \\ 4x + 0y + 0z \end{bmatrix}_{4\times 1}$$

$$= \begin{bmatrix} 7 & 1 & -1 \\ 2 & -6 & 6 \\ 3 & -3 & 3 \\ 4 & 0 & 0 \end{bmatrix}_{4\times 3} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times 1}$$

$$[T_2 \circ T_1] =$$

$$?\ [T_2 \circ T_1] \stackrel{?}{=} [T_2]\ ?\ [T_1]\ ?$$

Composition <=> matrix multiplication!

$\boxed{\text{Thm}}$ $$[T_2 \circ T_1] = [T_2] * [T_1]$$

Ex: $[T_1] = \begin{bmatrix} 1 & -1 & 1 \\ 3 & 1 & -1 \end{bmatrix}$ 2×3 $[T_2] = \begin{bmatrix} 1 & 2 \\ 5 & -1 \\ 3 & 0 \\ 1 & 1 \end{bmatrix}$ 4×2

$$\underbrace{[T_2]_{4\times 2} * [T_1]_{2\times 3}}_{=} = [\ ?\ ]_{4\times 3}$$

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 1 & 2 \\ 5 & -1 \\ 3 & 0 \\ 1 & 1 \end{bmatrix} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 & 1 \\ 3 & 1 & -1 \end{bmatrix} \end{array} = \begin{bmatrix} 7_{11} & 1_{12} & -1_{13} \\ 2 & -6 & 6 \\ 3 & -3 & 3 \\ 4_{41} & 0 & 0 \end{bmatrix}$$

def Matrix Multiplication = LT composition

$A=(a_{ij})_{m\times k}$ & $B=(b_{ij})_{k\times n}$

$$\underbrace{m\times k \quad \& \quad k\times n}_{=}$$

$A*B$ = matrix product of A & B

$= C_{m\times n} = (c_{ij})_{m\times n}$

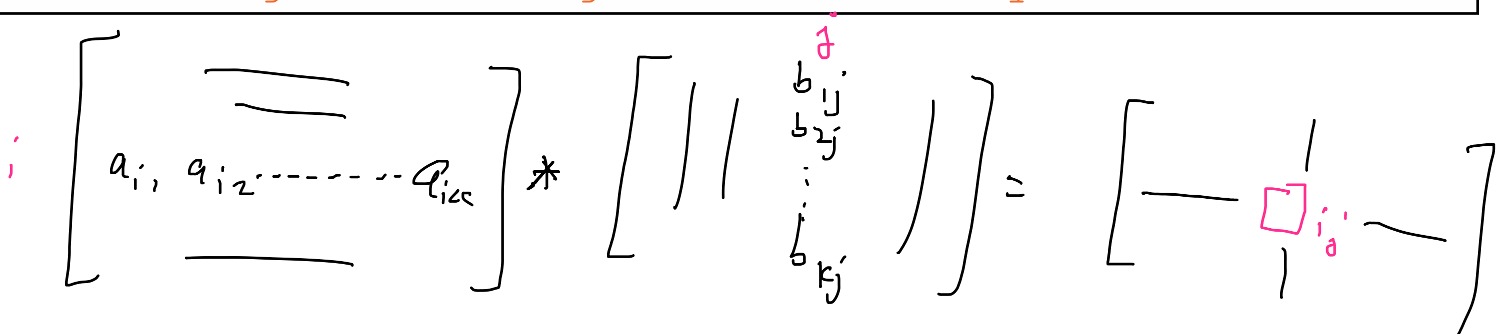
where $c_{ij} = (\text{row } i \text{ from } A)\circ(\text{column } j \text{ from } B)$

$$= [a_{i1}\ a_{i2}\ \cdots\cdots\ a_{ik}] \underset{\text{dot product}}{\circ} \begin{bmatrix} b_{1j}\\ b_{2j}\\ \vdots\\ b_{kj}\end{bmatrix}$$

$$= a_{i1}\cdot b_{1j} + a_{i2}\cdot b_{2j} + \cdots + a_{ik}\cdot b_{kj}$$

$$= \sum_{l=1}^{k} a_{il}\cdot b_{lj}$$

Note: this gives a single number/entry in A*B



Alternatives

$A*B = [A\vec{b}_1 \,|\, A\vec{b}_2 \,|\, \cdots \,|\, A\vec{b}_n]$ (here $\vec{b}_j$ col of B)

row A $= [\vec{a}_i \circ \vec{b}_j]$