2.3 Operations on Linear Transformations and Matrices

Definitions: If $\boxed{T_1 : \mathbb{R}^n \to \mathbb{R}^m}$ and $\boxed{T_2 : \mathbb{R}^n \to \mathbb{R}^m}$ are linear transformations, and $k \in \mathbb{R}$, then we can define:

Sum of T_1 & T_2

$$\boxed{T_1 + T_2 : \mathbb{R}^n \to \mathbb{R}^m,}$$
$$\boxed{T_1 - T_2 : \mathbb{R}^n \to \mathbb{R}^m,} \text{ and}$$

Scalar Product of k & T_1

$$\boxed{kT_1 : \mathbb{R}^n \to \mathbb{R}^m,}$$

as linear transformations, with actions given, respectively, by:

define

$$\boxed{(T_1 + T_2)(\vec{v}) = T_1(\vec{v}) + T_2(\vec{v}),}$$
$$(T_1 - T_2)(\vec{v}) = T_1(\vec{v}) - T_2(\vec{v}), \text{ and}$$

define

$$\boxed{(kT_1)(\vec{v}) = kT_1(\vec{v}).}$$

Arithmetic of Matrices

" size "

Definitions: If A and B are both $m \times n$ matrices, and k is any scalar, then we can define:

rows cols

$$\boxed{A + B}, A - B, \text{ and } \boxed{kA}$$

as $m \times n$ matrices with entries given by:

Note: Matrix Add/Sub need them to be SAME

what to do in each entry $1 \le i \le m$ $1 \le j \le n$

$$\boxed{(A + B)_{i,j} = (A)_{i,j} + (B)_{i,j},}$$

$$(A - B)_{i,j} = (A)_{i,j} - (B)_{i,j}, \quad \text{and}$$

$$(kA)_{i,j} = k(A)_{i,j}.$$

We call these the sum and difference of A and B, and the scalar multiple of A by k.

In particular, we can define the ***negative*** of a matrix, $\boxed{-A,}$ to be:

$$\boxed{-A = (-1)A} = (-a_{ij})_{m \times n}$$

with the property that:

$$A + (-A) = (-A) + A = 0_{m \times n}$$

$$\boxed{A + (-A) = [0]_{m \times n}}$$

Connection Between Linear Transformations and Matrices

Theorem: If $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^n \to \mathbb{R}^m$ are linear transformations, with matrices $[T_1]$ and $[T_2]$ respectively, and k is any scalar, then for any $\vec{v} \in \mathbb{R}^n$:

$$\boxed{\begin{aligned} (T_1 + T_2)(\vec{v}) &= ([T_1] + [T_2])\vec{v} \\ (T_1 - T_2)(\vec{v}) &= ([T_1] - [T_2])\vec{v} \\ (kT_1)(\vec{v}) &= (k[T_1])\vec{v} \end{aligned}}$$

Consequently, $T_1 + T_2$, $T_1 - T_2$ and kT_1 are linear transformations with matrices given by, respectively:

$$\boxed{\begin{aligned} [T_1 + T_2] &= [T_1] + [T_2], \\ [T_1 - T_2] &= [T_1] - [T_2], \text{ and} \\ [kT_1] &= k[T_1] \end{aligned}}$$

Compositions of Linear Transformations

Definition/Theorem: If $T_1 : \mathbb{R}^n \to \mathbb{R}^k$ and $T_2 : \mathbb{R}^k \to \mathbb{R}^m$ are linear transformations, then we can define their ***composition***:

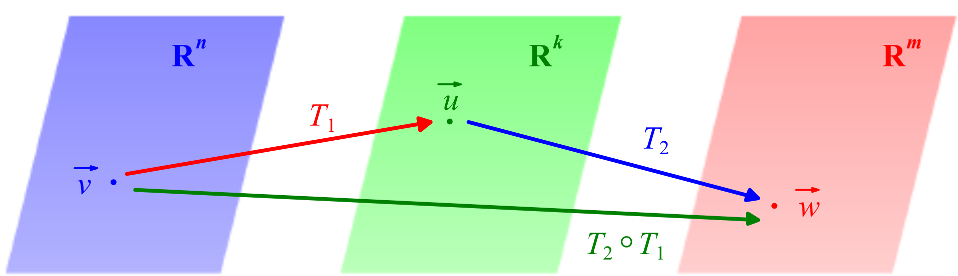
$$T_2 \circ T_1 : \mathbb{R}^n \to \mathbb{R}^m,$$

as a linear transformation, with action given as follows:

Suppose $\vec{u} \in \mathbb{R}^n$, $T_1(\vec{u}) = \vec{v} \in \mathbb{R}^k$, and $T_2(\vec{v}) = \vec{w} \in \mathbb{R}^m$. Then:

$$\boxed{(T_2 \circ T_1)(\vec{u}) = T_2(T_1(\vec{u}))} = T_2(\vec{v}) = \vec{w}.$$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "T_1"] \arrow[rr, bend right, "T_2\circ T_1"'] & \mathbb{R}^k \arrow[r, "T_2"] & \mathbb{R}^m \end{tikzcd}$$



General Matrix Products

Definition — Matrix Product:

If A is an $m \times k$ matrix, and B is a $k \times n$, then we can construct the $m \times n$ matrix AB, where:

$$\text{column } i \text{ of } AB = A \times (\text{column } i \text{ of } B)$$

In other words, if we write B in terms of its columns as:

$$B = \left[\vec{c}_1 \mid \vec{c}_2 \mid \cdots \mid \vec{c}_n \right]$$

then:

$$AB = \left[A\vec{c}_1 \mid A\vec{c}_2 \mid \cdots \mid A\vec{c}_n \right]$$

NOTE: see the extra notes for an alternative way to think about matrix multiplication.

NOTE: entry in i-th row, j-th column is: (row i of A) o (column j of B) [yes, this is a dot product!)

Linear Combinations of Linear Transformations and Matrices

Definition of a linear combination of linear transformations T_1, T_2, ..., T_k.

$$(c_1 T_1 + c_2 T_2 + \cdots + c_k T_k)(\vec{v})$$
$$= c_1 T_1(\vec{v}) + c_2 T_2(\vec{v}) + \cdots + c_k T_k(\vec{v})$$

Note: for this to make sense, need all of them to be from same input same to same output space. Namely, T_i : R^n --> R^m for all i=1,...,k

$$c_1 A_1 + c_2 A_2 + \cdots + c_k A_k.$$

Of course, by the Equivalence of LT and Mat Thm, anything we can do for LTs we can do for Mat.

$$[c_1 T_1 + c_2 T_2 + \cdots + c_k T_k]$$
$$= c_1 [T_1] + c_2 [T_2] + \cdots + c_k [T_k]$$

Not surprisingly, matrix of the LC of LTs equals the LC of Mats