2.4 Properties of Operations on Linear Transformations and Matrices

Goal: Show that matrix operations enjoy many $\boxed{\text{(but not all!!!)}}$ of the properties of the analogous operations on ordinary real numbers.

Properties of Matrix Addition and Scalar Multiplication

Theorem: If A, B and C are $m \times n$ matrices, and r and s are scalars, then the following properties hold:

Note: all must be SAME SIZE!

1. *The Commutative Property of Addition:*

$$A + B = B + A$$

2. *The Associative Property of Addition:*

$$A + (B + C) = (A + B) + C$$

3. *The "Left" Distributive Property:*

$$(r + s)A = rA + sA$$

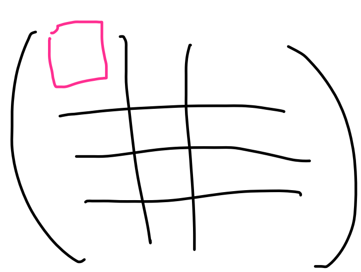
$r, s \in \mathbb{R}$

4. *The "Right" Distributive Property:*

$$r(A + B) = rA + rB$$

5. *The Associative Property of Scalar Multiplication:*

$$r(sA) = (rs)A = s(rA)$$



Properties of Matrix Multiplication

Theorem: If A and B are $m \times k$ matrices, C and D are $k \times n$ matrices, and r is a scalar, then the following properties hold:

1. ***The "Left" Distributive Property:***

$$\boxed{(A + B)C = AC + BC}$$

matrix
multiplication

2. ***The "Right" Distributive Property:***

$$\boxed{A(C + D) = AC + AD}$$

3. ***The Associative Property of Mixed (Scalar and Matrix) Products:***

$$\boxed{r(BC) = (rB)C = B(rC)}$$

The Associative Property of Matrix Multiplication

Theorem: If A is an $m \times p$ matrix, B is a $p \times q$ matrix, and C is a $q \times n$ matrix, then $\boxed{A(BC) = (AB)C.}$

Proof:

Both products $A(BC)$ and $(AB)C$ are $m \times n$ matrices.

Now, we have to show that both sides, pair-wise, have exactly the ***same entries***.

$\boxed{\text{Case 1: } C = \vec{x}, \text{ a } q \times 1 \text{ matrix.}}$

Write B as cols

$$\underline{B = \left[\vec{b}_1 \middle| \vec{b}_2 \middle| \ldots \middle| \vec{b}_q \right]}$$

Matrix Mult in form: A*(col of B) are the cols of A*B

$$\underline{AB = \left[A\vec{b}_1 \middle| A\vec{b}_2 \middle| \ldots \middle| A\vec{b}_q \right]}$$

$$\overset{C}{\downarrow}$$

$$C$$

$$\boxed{(AB)\vec{x}} = \left[A\vec{b}_1 \,\middle|\, A\vec{b}_2 \,\middle|\, \ldots\, A\vec{b}_q\right]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_q \end{bmatrix}$$

$$= x_1\left(A\vec{b}_1\right) + x_2\left(A\vec{b}_2\right) + \cdots + x_q\left(A\vec{b}_q\right)$$

Now, let us work on $A(B\vec{x})$:

$$C$$

$$\overset{C}{\downarrow}$$

$$B\vec{x} = \left[\vec{b}_1\ \vec{b}_2\ \ldots\ \vec{b}_q\right]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_q \end{bmatrix}$$

$$= x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_q\vec{b}_q$$

Recall: "Right Dist Prop" $(A+B)\vec{x} = A\vec{x} + B\vec{x}$

$$
\begin{aligned}
A(B\vec{x}) &= A\Big(\underbrace{x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_q\vec{b}_q}\Big) \\
&= A\Big(x_1\vec{b}_1\Big) + A\Big(x_2\vec{b}_2\Big) + \cdots + A\Big(x_q\vec{b}_q\Big) \\
&\qquad \text{(by the “Right” Distributive Property)} \\
&= \underbrace{x_1\Big(A\vec{b}_1\Big) + x_2\Big(A\vec{b}_2\Big) + \cdots + x_q\Big(A\vec{b}_q\Big)}
\end{aligned}
$$

This is same as (A*B)x!

Case 2: C is an arbitrary $q \times n$ matrix:

$$C = \left[\vec{c}_1 \middle| \vec{c}_2 \middle| \ldots \middle| \vec{c}_n\right]$$

By case 1: we know $(AB)\vec{c}_i = A(B\vec{c}_i)$

for every column $\vec{c}_i$.

Thus, column i of $(AB)C$ is exactly the same as that of $A(BC)$, and therefore $(AB)C = A(BC)$.

$\square$

The Matrix of a Composition

Theorem: If $T_1 : \mathbb{R}^n \to \mathbb{R}^k$ and $T_2 : \mathbb{R}^k \to \mathbb{R}^m$ are linear transformations, then:

$$\boxed{[T_2 \circ T_1] = [T_2] * [T_1]}$$

Important Result!

Pf Let $A = [T_1]_{k \times n}$, $B = [T_2]_{m \times k}$

so $B * A = [T_2] * [T_1]$

NTS $B * A = [T_2 \circ T_1]$

$\iff \forall \vec{x} \in \mathbb{R}^n$: $(B * A)\vec{x} = [T_2 \circ T_1]\vec{x}$.

Let $\vec{x} \in \mathbb{R}^n$ be arbitrary. $\vec{x} = \langle x_1, x_2, \ldots, x_n \rangle$

$(T_2 \circ T_1)(\vec{x}) = T_2(T_1(\vec{x}))$ (def of $T_2 \circ T_1$)

"

$[T_2 \circ T_1]\vec{x}$ $= T_2(A\vec{x})$ (by Thm in §2.1 Equiv of LT & Mat)

$= B(A\vec{x})$ (")

$= (B * A)\vec{x}$ (by Associativity of Matrix Mult)

Same

k-fold Compositions

If $T_1, T_2, \ldots, T_{k-1}, T_k$ are all linear transformations with the property that ***the codomain of*** T_i ***is the domain of*** T_{i+1}, for all $i = 1..k-1$, then we can inductively construct the k***–fold composition*** of these linear transformations by:

$$\begin{aligned} &(T_k \circ T_{k-1} \circ \cdots \circ T_2 \circ T_1)(\vec{v}) \\ &= T_k((T_{k-1} \circ \cdots \circ T_2 \circ T_1)(\vec{v})) \end{aligned}$$

$$[T_k \circ T_{k-1} \circ \cdots \circ T_2 \circ T_1] = [T_k][T_{k-1}]\cdots[T_2][T_1]$$

Powers of Square Matrices and Linear Operators

Theorem: The matrix product AA can be formed *if and only if* A is an $n \times n$ matrix. Analogously, the composition $T \circ T$ can be formed if and only if $T : \mathbb{R}^n \to \mathbb{R}^n$, i.e., T is an ***operator***.

Write AA as A^2 and $T \circ T$ as T^2.

Similarly, by induction, we will write:

$$A^k = A \cdot A^{k-1} = A \cdot A \cdot \cdots \cdot A, \text{ and}$$

$$T^k(\vec{v}) = T(T^{k-1}(\vec{v})) = T(T(\ldots T(\vec{v})))$$

Evaluating a Polynomial with a Matrix:

$$\underbrace{A_{n\times n} * A_{n\times n}}_{=} = A^2, \quad A^k = A * A * \cdots * A$$

Definition: If $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k$ is a polynomial with real coefficients, and $\boxed{A \text{ is any } n \times n \text{ matrix}}$, then we define the ***polynomial evaluation***, $p(A)$, by:

$$p(A) = c_0 I_n + c_1 A + c_2 A^2 + \cdots + c_k A^k.$$

I_n = identity matrix of size $n \times n$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & & & 0 \\ \vdots & 0 & \ddots & & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

Ex $p(x) = 3 + 2x + 4x^2$.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}$$

$$A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

$$p(A) = 3 I_2 + 2A + 4A^2$$

$$= 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 4\begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} + \begin{bmatrix} 28 & 40 \\ 60 & 88 \end{bmatrix} = \boxed{\begin{bmatrix} 33 & 44 \\ 66 & 99 \end{bmatrix}}$$

Multiplication by I_n and $0_{m\times n}$

Theorem: If A is any $m \times n$ matrix, then:

Important: properties of Identity matrices!

$$\boxed{AI_n = A} \quad \text{and} \quad \boxed{I_m A = A;}$$

$$\boxed{A0_{n\times p} = 0_{m\times p}} \quad \text{and} \quad \boxed{0_{q\times m}\underbrace{A}_{m\times n} = 0_{q\times n}.}$$

Important: properties of Zero matrices!

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}_{n\times n}}_{=} = \begin{bmatrix} a_{11} & a_{12} & \\ & & \end{bmatrix}$$

$$\underbrace{I_{m\times m} A_{m\times n}}_{=} = A$$

Danger Zone!

Matrix Multiplication shares many of the properties of real numbers R. BUT NOT ALL!

The Existence of Zero Divisors:

Definition: Two $n \times n$ matrices A and B with the property that $\boxed{AB = 0_{n\times n}}$, but ***neither*** A nor B is $0_{n\times n}$ are called ***zero divisors***.

In other words, The Zero Factors Theorem does not hold for matrices.

$$\overset{A}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} * \overset{B}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\boxed{AB \neq BA \text{ Most of the Time!}}$

Matrix multiplication, in general, is NOT commutative!

Remember: Matrix multiplication doesn't have all the properties of regular multiplication in R.

Matrix Mult: fails Zero-Factors Theorem! Since we can find matries A*B=[0] but A is not [0] and B is not [0]!

Matrix Mult: IS NOT COMMUTATIVE! Since A*B = B*A is almost never true. Moreover, A*B is not even always defined!

A Linear Transformation is Uniquely Determined by any Basis

Theorem: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, and $\boxed{B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}}$ is any ***basis*** for $\mathbb{R}^n$, then the action of T is uniquely determined by the vectors $\boxed{\{T(\vec{v}_1), T(\vec{v}_2), \dots, T(\vec{v}_n)\}}$ from $\mathbb{R}^m$.

$c_1, c_2, \dots, c_n$ are all unique

More specifically, if $\vec{v} \in \mathbb{R}^n$ and $\vec{v}$ is expressed (uniquely) as $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$, then:

$$\boxed{T(\vec{v}) = c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + \cdots + c_n T(\vec{v}_n).}$$

Idea: $T(\vec{v}) = \begin{bmatrix} T(\vec{v}_1) & \cdots & T(\vec{v}_n) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

"in the right basis"

$\vec{x} = \langle c_1, c_2, \dots, c_n \rangle$

Don't take this literally right now. We will come back to this topic later. The idea is that we want to find the "best basis" that matches the properties of a LT T. This can mean several things. For example, we want to find a basis B that makes [T] the "simples". Again, we'll revist this idea at a later time.