2.5 One-to-One Transformations and Onto Transformations

The Kernel and Range of a Linear Transformation

Definition: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, we define the ***kernel*** of T as the set:

$$\boxed{ker(T) = \left\{ \vec{v} \in \mathbb{R}^n \,\middle|\, \boxed{T(\vec{v}) = \vec{0}_m} \right\} \subset \mathbb{R}^n.}$$

Similarly, we define the ***range*** of T as the set:

$$\boxed{range(T) = \left\{ \vec{w} \in \mathbb{R}^m \,\middle|\, \boxed{\vec{w} = T(\vec{v})} \text{ for some } \vec{v} \in \mathbb{R}^n \right\} \subset \mathbb{R}^m.}$$

We emphasize that $ker(T)$ is from $\mathbb{R}^n$, and $range(T)$ is from $\mathbb{R}^m$.

Recall: Fundamental subspaces of a matrix A. There's 4 of them, but two of them are NS(A) and CS(A).

Fundamental subspaces of a Linear Transformation! Notice that ker(T) corresponds to NS(A) and Range(T) corresponds to CS(A).

Theorem: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a ***linear transformation***, then:

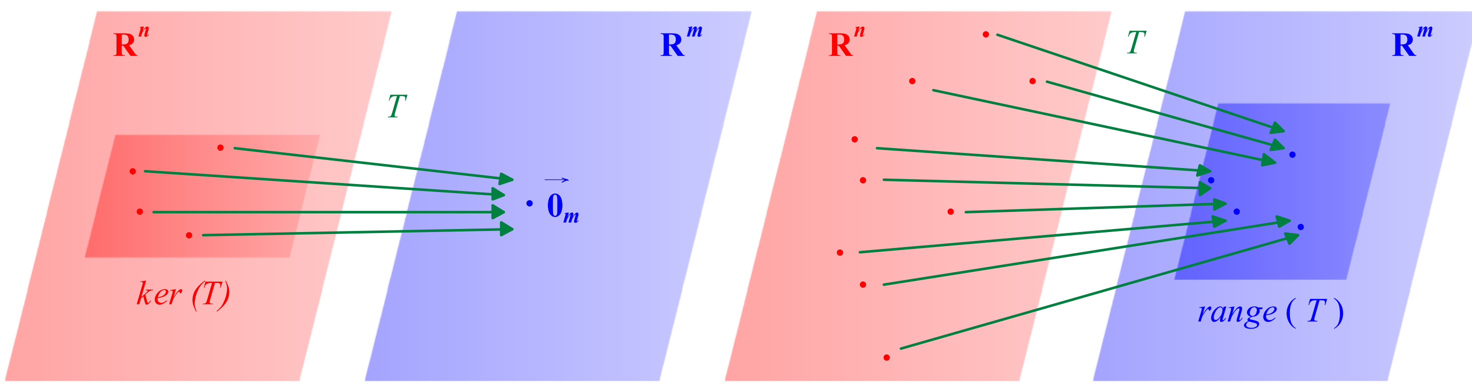
$$\boxed{ker(T) = nullspace([T]) \trianglelefteq \mathbb{R}^n}, \text{ and}$$

$$\boxed{range(T) = colspace([T]) \trianglelefteq \mathbb{R}^m.}$$

We call the dimension of $ker(T)$ the ***nullity* of T**, written $nullity(T)$. Similarly, we call the dimension of $range(T)$ the ***rank*** of T, written $rank(T)$. Thus:

$$nullity(T) = dim(nullspace([T])) = nullity([T]), \text{ and}$$

$$rank(T) = dim(colspace([T])) = rank([T]).$$



The Kernel of T The Range of T

So these are subspaces! These are important so make sure to study these carefully!

The Dimension Theorem for Linear Transformations

Theorem: Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Then:

$$\boxed{rank(T) + nullity(T) = n}$$

Pf By the dimension theorem for matrices:

$$rank([T]) + nullity([T]) = n$$

$$rank(T) + nullity(T) = n. \quad \square$$

Remark: It's hard to overstate the important of the dimension theorem for matrices we proved in Ch 1! It's one of the main tools of this entire chapter.

One-to-One Transformations

Definition: We say that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is ***one-to-one*** (or ***injective***) if the image of two different vectors from the domain are different vectors of the codomain:

$$\text{If } \vec{v}_1 \neq \vec{v}_2 \text{ then } T(\vec{v}_1) \neq T(\vec{v}_2).$$

We also say that T is an ***injection*** or an ***embedding***.

Theorem: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is ***one-to-one*** ***if and only if*** the only way two vectors from the domain have the same image in the codomain is for them to be the same vector to begin with:

more helpful way:

$$\boxed{\text{If } T(\vec{v}_1) = T(\vec{v}_2) \text{ then } \vec{v}_1 = \vec{v}_2.}$$

In other words, the ***only solution*** to $T(\vec{v}_1) = T(\vec{v}_2)$ is $\vec{v}_1 = \vec{v}_2$.

Pf just converse of def of 1-1 above. □

Easy but important!

Study this proof! Good test Q!

Theorem — The Kernel Test for Injectivity:

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is *one-to-one* *if and only if*:

$$\boxed{ker(T) = \left\{\vec{0}_n\right\}}.$$

($\Rightarrow$) We are given that T is one-to-one.

We must show that $ker(T) = \left\{\vec{0}_n\right\}$.

Suppose $\vec{v} \in ker(T)$. By def of kernel, we have: $T(\vec{v}) = \vec{0}$.
We have $T(\vec{0}) = \vec{0}$, since T is a LT. so:

$T(\vec{v}) = T(\vec{0})$ so b/c T is 1-1 $\Longrightarrow$ $\vec{v} = \vec{0}$.

($\Leftarrow$) We are given that $ker(T) = \left\{\vec{0}_n\right\}$.

We must show that T is one-to-one.

So suppose $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$, and $T(\vec{v}_1) = T(\vec{v}_2)$. NTS: $\vec{v}_1 = \vec{v}_2$.

$T(\vec{v}_1) = T(\vec{v}_2) \Longleftrightarrow T(\vec{v}_1) - T(\vec{v}_2) = \vec{0}_m$

$\Longleftrightarrow T(\vec{v}_1 - \vec{v}_2) = \vec{0}_m$ (b/c T is a LT
- Add prop &
- Scal Product Prop)

This says: $\vec{v}_1 - \vec{v}_2 \in ker(T)$.

By assumption, $\vec{v}_1 - \vec{v}_2 = \vec{0}_n$. Thus, $\vec{v}_1 = \vec{v}_2$.

$\square$

Example: Suppose $T_1, T_2 : \mathbb{R}^3 \to \mathbb{R}^4$ are given by the following matrices with the corresponding rrefs. Describe the kernel of each, decide if either is one-to-one, and verify the Dimension Theorem for both.

$$\ker(T_1) = NS([T_1]) = NS(R) = \text{Span}\left(\left\{\begin{bmatrix}3\\1\\0\end{bmatrix}\right\}\right)$$

Since $\ker(T_1) \neq \{\vec{0}\}$ T_1 is not 1-1.

$$[T_1] = \begin{bmatrix} 1 & -3 & 4 \\ 2 & -6 & 9 \\ 5 & -15 & 4 \\ -3 & 9 & -7 \end{bmatrix}_{4\times 3},$$

$$\text{with rref } \underbrace{R_1 = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{R_1\vec{x} = \vec{0}}$$

leading: x_1, x_3

free: x_2

$x_1 - 3x_2 = 0 \to x_1 = 3x_2$

$x_3 = 0 \to x_3 = 0$

$$\vec{x} = \begin{bmatrix} 3x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$NS(R_1) = \text{Span}\left(\left\{\begin{bmatrix}3\\1\\0\end{bmatrix}\right\}\right)$. $\text{Nullity}(T_1) = 1$.

all LI

$$[T_2] = \begin{bmatrix} 1 & -2 & 4 \\ 2 & -6 & 9 \\ 5 & -15 & 4 \\ -3 & 9 & -7 \end{bmatrix},$$

4x3

$$\text{with rref} \quad R_2 = \begin{bmatrix} \boxed{1 \quad 0 \quad 0} \\ \boxed{0 \quad 1 \quad 0} \\ \boxed{0 \quad 0 \quad 1} \\ 0 \quad 0 \quad 0 \end{bmatrix}. \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$NS(R_2) = NS([T_2]) = \ker(T_2)$

Describe the kernel and range of each, decide if either is one-to-one, and verify the Dimension Theorem for both.

$x_1 = 0$

$x_2 = 0$

$x_3 = 0$

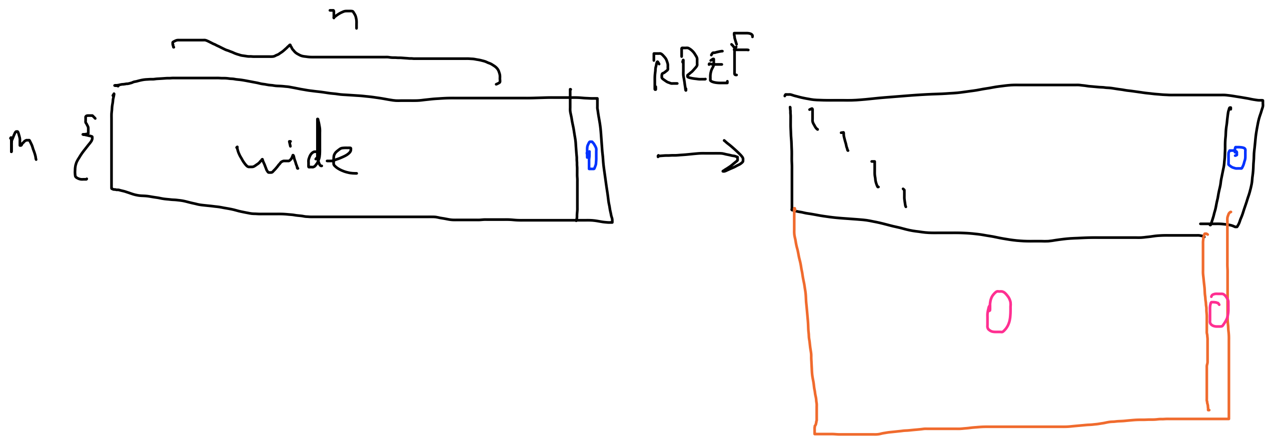
$$\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so $\ker(T_2) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

so T_2 is 1-1

Theorem: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is ***not*** one-to-one if $n > m$.

<u>Pf</u>



can add rows of zeros

so $NS([T]) =$ has ∞ly many solutions

$\quad\;\, \parallel$

$\quad \mathrm{Ker}(T)$

so $\mathrm{Ker}(T) \neq \{\vec{0}\}$. So T is <u>not</u> 1-1. $\square$

Onto Linear Transformations

$Range(T) \trianglelefteq \mathbb{R}^m$

Definition: We say that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is ***onto*** or ***surjective*** if:

ie onto iff range is as big as possible.

$$range(T) = \mathbb{R}^m.$$

We also say that T is a ***surjection*** or a ***covering*** (because T hits all the vectors of $\mathbb{R}^m$).

$Range(T) = \mathbb{R}^m$: $\forall \vec{w} \in \mathbb{R}^m \ \exists \vec{v} \in \mathbb{R}^n$ so that $\vec{w} = T(\vec{v})$.

Theorem: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is ***onto*** ***if and only if*** $rank(T) = m$.

Pf $rank(T) = dim\big(cs([T])\big) = dim\big(Range(T)\big)$

- if T is onto then $Range(T) = \mathbb{R}^m$ so $dim(\mathbb{R}^m) = m$ so $rank(T) = m$.
- if $rank(T) = m$, then $dim(Range(T)) = m$ but $\underbrace{Range(T)}_{m \text{ LI basis.}} \trianglelefteq \mathbb{R}^m$

then $\mathbb{R}^m = Range(T)$ so T is onto by def. $\square$

Example: Suppose $T_1, T_2 : \mathbb{R}^3 \to \mathbb{R}^2$ are given by the following matrices with the corresponding rrefs. Describe the kernel and range of each, decide if either is one-to-one, and/or onto, and verify the Dimension Theorem for both.

rank + nullity = 1+2 = 3 = n ← input $\mathbb{R}^n$

$$[T_1] = \begin{bmatrix} -2 & -8 & 6 \\ 1 & 4 & -3 \end{bmatrix}, \quad \text{with rref} \quad R_1 = \begin{bmatrix} 1 & 4 & -3 \\ 0 & 0 & 0 \end{bmatrix}.$$

dim(range(T_1)) = 1 $\boxed{\text{rank}(T_1) = 1}$

dim(ker(T_1)) = 2 $\boxed{\text{nullity}(T_1) = 2}$

leading : 1 → CS

free : 2 → NS

T_1 is not 1-1. T_1 is not onto

$$[T_2] = \begin{bmatrix} -2 & -8 & 7 \\ 1 & 4 & -3 \end{bmatrix}, \quad \text{with rref} \quad R_2 = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

dim(range(T_2)) = 2 ←

dim(ker(T_2)) = 1

leading : 2 → CS

free : 1 → NS

rank(T_2) + nullity(T_2) = n

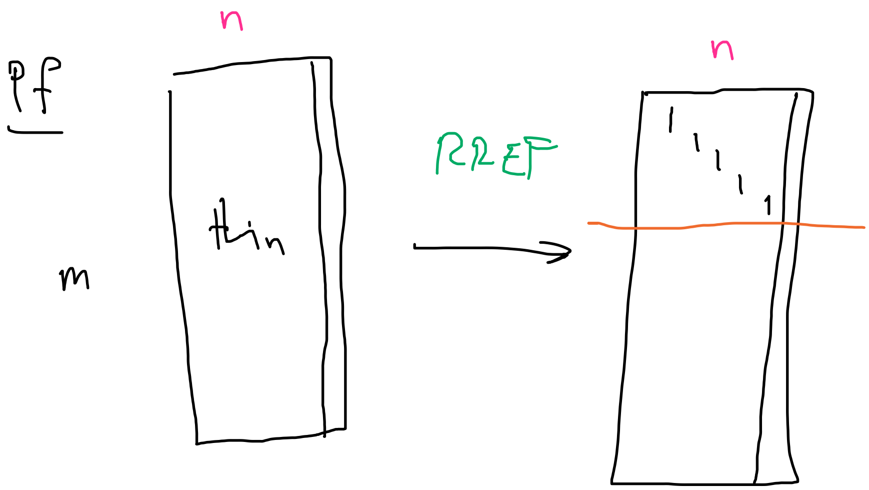
2 + 1 = 3

T_2 is not 1-1.

T_2 is onto

Theorem: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is $\boxed{\textit{not} \text{ onto}}$ if $\boxed{n < m}$.

Pf



max # of leading 1s $= n < m$

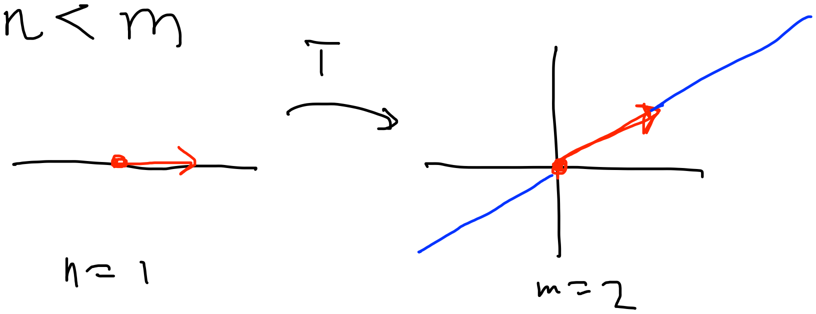
$\underbrace{\dim(CS)}_{\dim(\mathrm{Range}(T))} = n < m$

so $\mathrm{Range}(T) \neq \mathbb{R}^m$ $\square$

Helpful "cartoons" to keep in mind:

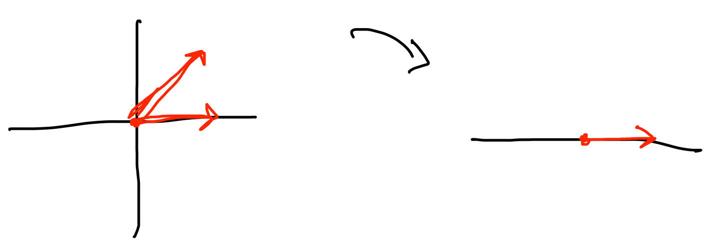
$T : \mathbb{R}^n \to \mathbb{R}^m$

- $n < m$



$\boxed{\text{not onto}}$

- $m > n$



$\boxed{\text{not 1-1}}$

Using the RREF of the Matrix of T

Theorem: Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, and R is the rref of $[T]$.

Then:

iff $\ker(T) = \{\vec{0}\}$ iff

1. T is ***one-to-one*** ***if and only if*** R does ***not*** have any free variables.

2. T is ***onto*** ***if and only if*** R does ***not*** have any row consisting only of zeroes. if and only if R has all leading 1s

iff $\text{Range}(T) = \mathbb{R}^m$ iff

Theorem — Equivalent Properties for $\boxed{\text{Full-Rank}}$ Linear Transformations:

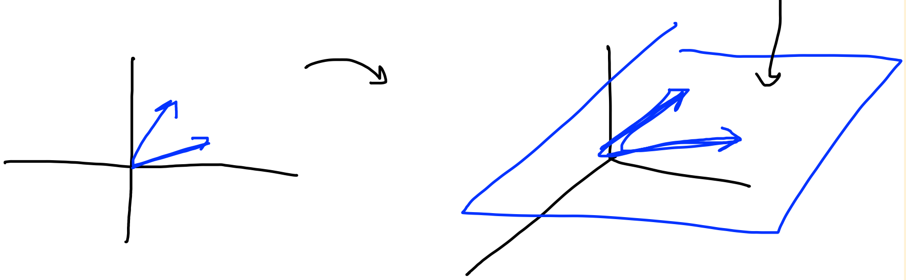
Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Then:

1. if $m < n$: T is full-rank ***if only only if*** T is ***onto***.
2. if $m = n$: T is full-rank ***if and only if*** T is ***both one-to-one and onto***.
3. if $m > n$: T is full-rank ***if and only if*** T is ***one-to-one***.

Proof: Exercise. $\boxed{\text{Dim Thm}}$ $\text{rank}(T) + \text{nullity}(T) = n$

$\boxed{\underline{\text{Full rank}}: \ \text{rank}(T) = n}$

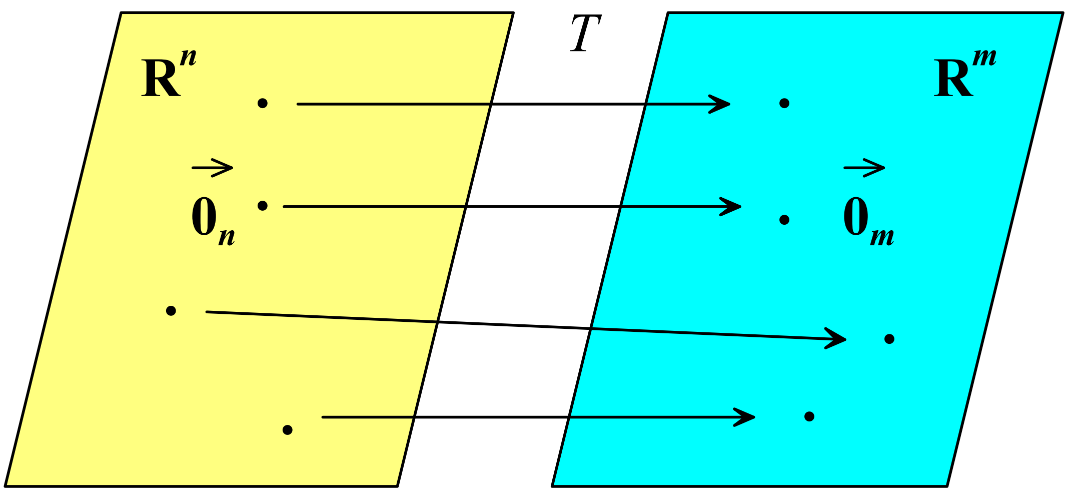
3 $n=2$, $m=3$ Full Rank



Carefull: full rank is not the same as onto! By Dimension Theorem, it just means that the rank(T)=n. So if we have n LI vectors in R^n, then full-rank means T(v_1), ..., T(v_n) are also LI in R^m

A Recap of The One-to-One and Onto Properties

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is ***one-to-one*** if and only if $ker(T) = \left\{\vec{0}_n\right\}$. This means that if $\vec{v} \in \mathbb{R}^n$ is any other vector but $\vec{0}_n$, then $T(\vec{v}) \neq \vec{0}_m$.

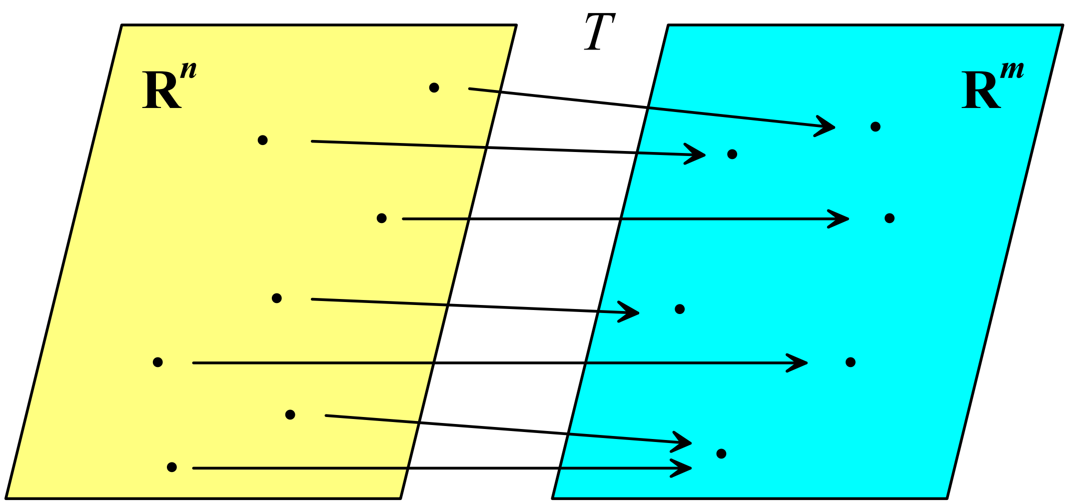


T is ***one-to-one*** if and only if:

$$ker(T) = \left\{\vec{0}_n\right\}$$

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is ***onto*** if and only if $range(T) = \mathbb{R}^m$.

This means that for any vector $\vec{w} \in \mathbb{R}^m$, we can find at least one vector $\vec{v} \in \mathbb{R}^n$ such that $T(\vec{v}) = \vec{w}$. We remark that more than one such vector $\vec{v}$ could exist for every $\vec{w}$. This also means that $rank(T) = m$.



T is ***onto*** if and only if:

$$range(T) = \mathbb{R}^m$$

Anything Can Happen:

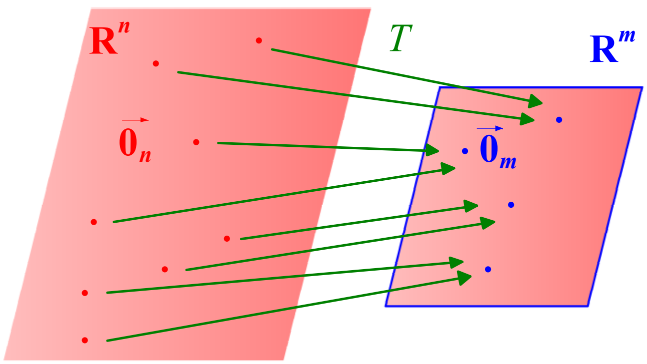
Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$.

If we ***don't*** know anything about n or m, then T can be:

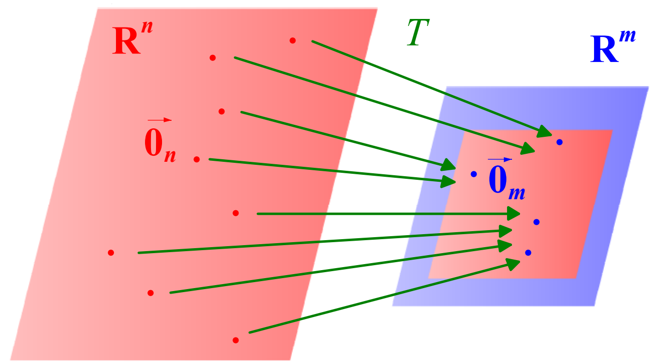
- one-to-one but not onto;
- onto but not one-to-one;
- ***neither*** one-to-one nor onto;
- ***both*** one-to-one and onto.

However, if we ***knew*** that:

- $n > m$, then T is automatically ***not one-to-one***; however, T can be onto, or not onto.



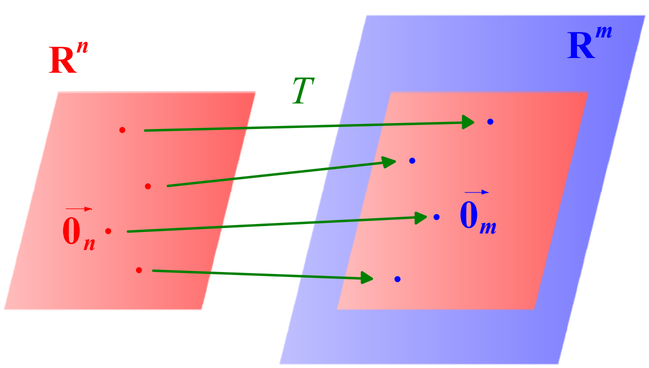
T is onto, but not one-to-one
rank $(T) = m$; T is full rank



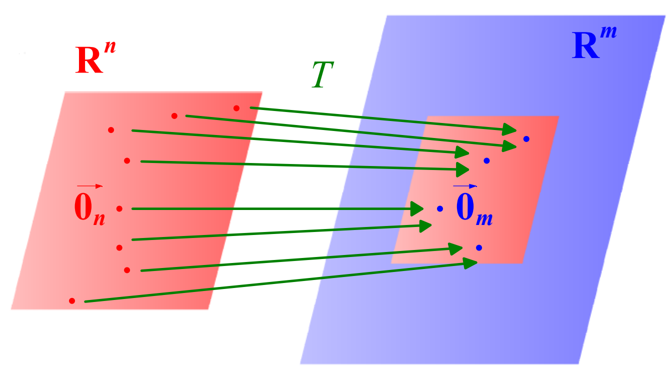
T is neither onto nor one-to-one
rank $(T) < m$

$\boldsymbol{n > m}$

- $n < m$, then T is automatically ***not onto***; however, T can be one-to-one, or not one-to-one.



T is one-to-one, but not onto
rank $(T) = n$; T is full rank



T is neither one-to-one nor onto
rank $(T) < n$

$\boldsymbol{n < m}$