$\boxed{2.6}$

$\boxed{\text{Thm}}$ T invertible $\boxed{\text{iff}}$ $\exists!$ operator S "unique"

so that

$$\boxed{S \circ T = Id_{\mathbb{R}^n}}$$

$$\& \quad \boxed{T \circ S = Id_{\mathbb{R}^n}}$$

Call such an operator S the inverse of T & write

$$S = T^{-1}.$$

If S exists: $S(\vec{w}) = \vec{v}$ when $\vec{w} = T(\vec{v})$

Pf

($\Longrightarrow$) Assume T is invertible.

$\Rightarrow$ T is onto

$\Rightarrow$ T is 1-1.

Let $\vec{w} \in \mathbb{R}^n$ be arbitrary. We're going to define an operator $S: \mathbb{R}^n \to \mathbb{R}^n$ w/ desired prop.

Since T is onto: $\exists\, \vec{v} \in \mathbb{R}^n$: $T(\vec{v}) = \vec{w}$.

Define

$$S(\vec{w}) = \vec{v}.$$

so S defined on all of $\mathbb{R}^n$.

Note: well-defined is the name for checking that S is indeed a function, i.e. "it passes the VLT"

Check well-defined: if $\vec{w}_1 = \vec{w}_2$ then $S(\vec{w}_1) = S(\vec{w}_2)$

True b/c T is 1-1. $\exists\, \vec{v}_1, \vec{v}_2$ so that

$T(\vec{v}_1) = \vec{w}_1$, $T(\vec{v}_2) = \vec{w}_2$ if $\vec{w}_1 = \vec{w}_2$

then $T(\vec{v}_1) = T(\vec{v}_2)$ so $\vec{v}_1 = \vec{v}_2$ b/c T is 1-1.

Note

- $S(T(\vec{v})) = S(\vec{w}) = \vec{v}$.

  $\hookrightarrow$ $\boxed{S \circ T = Id_{\mathbb{R}^n}}$

- $T(S(\vec{w})) = T(\vec{v}) = \vec{w}$.

  $\hookrightarrow$ $\boxed{T \circ S = Id_{\mathbb{R}^n}}$

- **NTS** S is LT.

  - **Add Prop** $\vec{w}_1, \vec{w}_2 \in \mathbb{R}^n$

    $S(\vec{w}_1 + \vec{w}_2)$:

  - $\exists! \ \vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$: $T(\vec{v}_1) = \vec{w}_1$ (T onto)

    $T(\vec{v}_2) = \vec{w}_2$

  T is LT:

$$T(v_1) + T(v_2) = T(\vec{v}_1 + \vec{v}_2)$$

So $\vec{w}_1 + \vec{w}_2 = T(\vec{v}_1) + T(\vec{v}_2) = T(\vec{v}_1 + \vec{v}_2)$.

So

$$\begin{aligned} S(\vec{w}_1 + \vec{w}_2) &= S\left(T(\vec{v}_1 + \vec{v}_2)\right) \\ &= \underbrace{(S \circ T)}_{Id}(\vec{v}_1 + \vec{v}_2) \\ &= \vec{v}_1 + \vec{v}_2 \\ &= S(\vec{w}_1) + S(\vec{w}_2) \quad (\text{def of } S) \end{aligned}$$

- Scal Prod. Prop: exercise

- - - - - - - - - - - - -

($\Leftarrow$) Assume S exists, LT.

<u>NTS</u>: T is 1-1 & onto.

- <u>T is 1-1</u>: Assume $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$.

  Assume $T(\vec{v}_1) = T(\vec{v}_2)$

  <u>NTS</u> $\vec{v}_1 = \vec{v}_2$.

$$S(T(\vec{v}_1)) = S(T(\vec{v}_2))$$

$$(S \circ T)(\vec{v}_1) = (S \circ T)(\vec{v}_2)$$

$$Id_{\mathbb{R}^n}(\vec{v}_1) = Id_{\mathbb{R}^n}(\vec{v}_2)$$

$$\vec{v}_1 = \vec{v}_2.$$

- <u>T is onto</u>: Let $\vec{w} \in \mathbb{R}^n$ be arbitrary.

  <u>NTS</u> $\exists \vec{v} \in \mathbb{R}^n : T(\vec{v}) = \vec{w}$

Take $\vec{v} = S(\vec{w})$. Then

By assumption, S exits and can map w to something in R^n. Call this v.

$$T(\vec{v}) = T(S(\vec{w}))$$
$$= (T \circ S)(\vec{w})$$
$$= Id_{\mathbb{R}^n}(\vec{w})$$
$$= \vec{w}.$$

$\square$

$$(A \mid I_n) \longrightarrow (I_n \mid A^{-1})$$

why? $A * \underbrace{A^{-1}}_{[\vec{x}_1 | \vec{x}_2 | \cdots | \vec{x}_n]} = I_n$

$$[A\vec{x}_1 \mid A\vec{x}_2 \mid \cdots \mid A\vec{x}_n] = [I_n]$$

Solving n SOEs:

$$A\vec{x}_1 = \begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix} \longleftrightarrow \left(A\vec{x}_1 \,\middle|\, \vec{e}_1\right)$$

$$\& \quad A\vec{x}_2 = \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix} \longleftrightarrow \left(A\vec{x}_2 \,\middle|\, \vec{e}_2\right)$$

$$\& \quad \vdots$$

$$A\vec{x}_n = \begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix} \longleftrightarrow \left(A\vec{x}_n \,\middle|\, \vec{e}_n\right)$$

$$\left[\, A \,\middle|\, I_n \,\right].$$