2.6 Invertible Operators and Matrices

In Algebra, we first require that a function:

$$f : D \to R,$$

with domain D and range R, is ***one-to-one*** on D before we find its inverse. If so:

$$f^{-1} : R \to D, \text{ where}$$

$$f^{-1}(y) = x \quad \textit{\textbf{if and only if}} \quad f(x) = y.$$

f and f^{-1} also possess the ***cancellation properties***:

$$\boxed{\begin{aligned} &f^{-1}(f(x)) = f^{-1}(y) = x \text{ for all } x \in D, \\ &f(f^{-1}(y)) = f(x) = y \text{ for all } y \in R. \end{aligned}} \quad \textbf{\textit{and}}$$

I prefer to call these the
"Inverse Properties"

Definition: We say that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is ***invertible*** if and only if T is ***both*** one-to-one and onto.

We also say equivalently that T is ***~~bijective~~***, T is a ***~~bijection~~*** or T is an ***isomorphism.***

Note: I prefer to reserve the term "bijective" for general functions that are 1-1 and onto. Of course, LTs are special functions so you can use the terms. Isomorphism is ok to use since this term applies to LTs from vector spaces (as we'll learn in Ch 3).

Theorem: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is invertible, then $\boxed{n = m}$.

Cor If T is invertible, then T is a linear operator.

Pf T is invertible

$\Rightarrow$ T is 1-1 : $\Rightarrow$ $\ker(T) = \{\vec{0}\}$

& onto : $\Rightarrow$ $\text{Range}(T) = \mathbb{R}^m$

- $\text{rank}(T) = \dim(\text{Range}(T)) = m$
- $\text{nullity}(T) = \dim(\ker(T)) = 0$

So $\boxed{\text{Dim Thm:}}$ $\text{rank}(T) + \text{nullity}(T) = n$

What else?!?

$m + 0 = n$

$m = n.$ $\square$

Says $\boxed{T : \mathbb{R}^n \to \mathbb{R}^n}$ invertible operator

Theorem: A linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible ***if and only if*** we can find another ***unique*** linear operator, $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$, the ***inverse operator*** for T, such that if $\vec{v} \in \mathbb{R}^n$ and $T(\vec{v}) = \vec{w}$, then we define:

$$\begin{tikzcd} \vec{v} \in \mathbb{R}^n \arrow[r, "T"] & \vec{w} \in \mathbb{R}^n \arrow[l, bend left, "T^{-1}"] \end{tikzcd}$$

$$T^{-1}(\vec{w}) = \vec{v},$$

and thus: Inverse Properties

$$\boxed{(T^{-1} \circ T)(\vec{v}) = \vec{v}} \quad \text{and} \quad \boxed{(T \circ T^{-1})(\vec{w}) = \vec{w}.}$$

In other words:

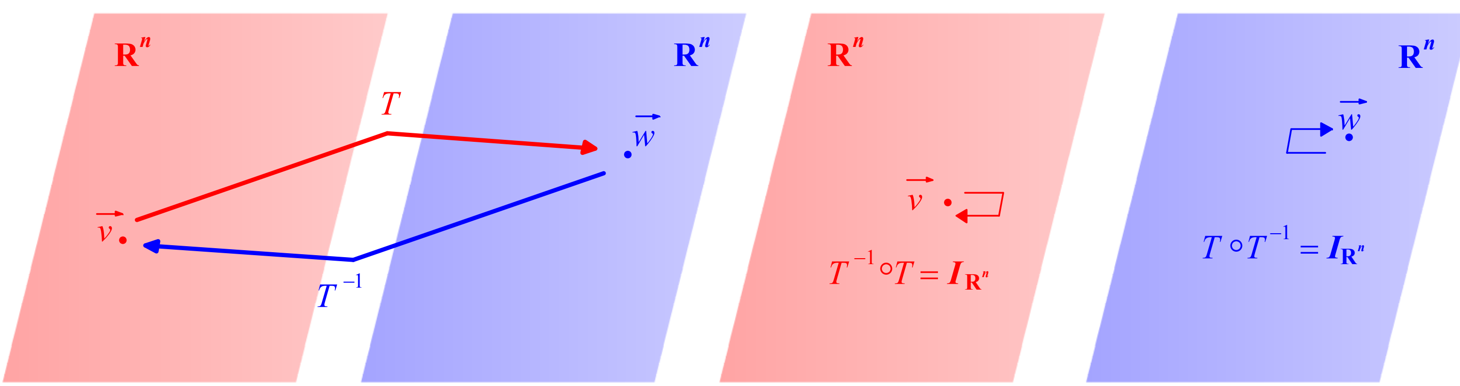
$$\boxed{T^{-1} \circ T = T \circ T^{-1} = I_{\mathbb{R}^n},}$$

$I_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$

$\vec{v} \mapsto \vec{v}$

the ***identity operator*** on $\mathbb{R}^n$.

Furthermore, if T is invertible, then T^{-1} is also invertible, and $\boxed{(T^{-1})^{-1} = T.}$ Thus, we can say that T and T^{-1} are ***inverses of each other***.



The Composition of T with T^{-1}

$$T^{-1} \circ T = I_{\mathbb{R}^n} = T \circ T^{-1}$$

Invertible Matrices

Definition: An $\boxed{n \times n}$ matrix A is ***invertible*** if and only if the linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ corresponding to $A = [T]$ is an invertible operator. In other words, the operator defined by:

$$T(\vec{v}) = A\vec{v},$$

for all $\vec{v} \in \mathbb{R}^n$, is an invertible operator on $\mathbb{R}^n$.

Theorem/Definition: An $n \times n$ matrix $\boxed{A \text{ is invertible}}$ ***if and only if*** we can find another $n \times n$ matrix B such that:

$$\boxed{AB = I_n} \cancel{=} BA. \quad \& \quad \boxed{BA = I_n}$$

We call B the ***inverse matrix*** of A, and denote it by $\boxed{A^{-1}}$.

If A is invertible, then the inverse matrix $\boxed{A^{-1}}$ is likewise invertible, and:

$$\boxed{(A^{-1})^{-1} = A.}$$

In other words, $B^{-1} = A$. Thus, we can say that A and A^{-1} are ***inverses of each other.***

Pf

($\Rightarrow$) A invertible $\Rightarrow$ $T: \mathbb{R}^n \to \mathbb{R}^n$ is invertible w/ [T]=A

$\Rightarrow$ $T^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ exists & LT & 1-1 & onto

$\Rightarrow$ our last class:

- $[Id_{\mathbb{R}^n}] = [T^{-1} \circ T] = [T^{-1}] * [T]$

Thus, set B=[T^(-1)]

$[T^{-1}] * [T] = I_n$ $\quad$ $(B * A = I_n)$

- $[Id_{\mathbb{R}^n}] = [T \circ T^{-1}] = [T] * [T^{-1}]$

$[T] * [T^{-1}] = I_n$ $\quad$ $(A * B = I_n)$

Let $B = [T^{-1}]$.

($\Leftarrow$) exercise. $\square$

Theorem: If an $n \times n$ matrix A is invertible, then its inverse matrix B is ***unique***. This means that if B and C both satisfy the equations:

$$AB = I_n = BA \quad \text{and} \quad AC = I_n = CA,$$

then $B = C$.

Pf

$$\begin{aligned} C = I_n * C &= (B * A) * C \\ &= B * (A * C) \qquad \text{(ass.)} \\ &= B * I_n \\ &= B. \end{aligned}$$

$\square$

Nice proof -- good test Q ;-)

Theorem: Suppose that:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then A is invertible *if and only if* $ad - bc \neq 0$, in which case:

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Pf in book. $\square$

Finding A^{-1} efficiently: GJR

$A_{n \times n}$. Find $A^{-1}_{n \times n}$. n^2 unknowns in A^{-1}

$A * A^{-1} = I_n$ n^2 eq in n^2 unknowns

Method

$$(A \mid I_n) \xrightarrow{GJR} (I_n \mid A^{-1})$$

"augmented matrix"

Example:

$$A = \begin{bmatrix} -3 & -5 \\ 5 & 7 \end{bmatrix}.$$

Example:

$$A = \begin{bmatrix} 3 & -7 \\ 12 & -28 \end{bmatrix}.$$

The Matrix of T^{-1}

Theorem: A linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible ***if and only if*** $A = [T]$ is an invertible $n \times n$ matrix. If this is the case, then:

$$\boxed{[T^{-1}] = A^{-1} = [T]^{-1}.}$$

Pf T is invertible $\iff$ A is invertible by def.

So only NTS: [T^(-1)] = [T]^(-1)

T is invertible T^{-1} exists :

We know: $\boxed{T^{-1} \circ T = Id_{\mathbb{R}^n}}$ & $\boxed{T \circ T^{-1} = Id_{\mathbb{R}^n}.}$

We know: $[T^{-1} \circ T] = [Id_{\mathbb{R}^n}]$ & $[T \circ T^{-1}] = [Id_{\mathbb{R}^n}]$

We know: $[T^{-1}] * [T] = I_n$ & $[T] * [T^{-1}] = I_n$

So: $\boxed{[T^{-1}] * A = I_n}$ & $\boxed{A * [T^{-1}] = I_n}$

Shows $A^{-1} = [T^{-1}]$. But use Uniqueness of Mat Inverse to get: $B = A^{-1} = [T^{-1}]$.

So $A^{-1} = [T]^{-1} = [T^{-1}]$. $\square$

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by:

$$T(\langle x, y\rangle) = \langle \boxed{3x + 7y}, \boxed{2x - 6y}\rangle.$$

$$[T] = \begin{bmatrix} 3 & 7 \\ 2 & -6 \end{bmatrix}_{2\times 2} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$[T^{-1}] = [T]^{-1} = \frac{1}{\Delta}\begin{bmatrix} -6 & -7 \\ -2 & 3 \end{bmatrix} = \frac{-1}{32}\begin{bmatrix} -6 & -7 \\ -2 & 3 \end{bmatrix}$$

where $\Delta = ad - bc$ $\qquad = \frac{1}{32}\begin{bmatrix} 6 & 7 \\ 2 & -3 \end{bmatrix}$

$\Delta = -18 - 14 = -32$

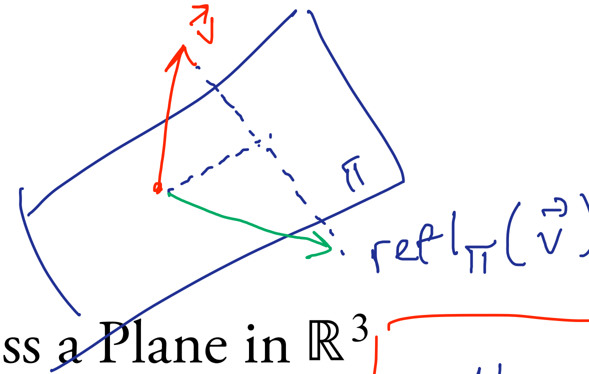
Check

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [T]^{-1} * [T] = \begin{bmatrix} 3 & 7 \\ 2 & -6 \end{bmatrix} * \frac{1}{32}\begin{bmatrix} 6 & 7 \\ 2 & -3 \end{bmatrix}$$

$$= \frac{1}{32}\begin{bmatrix} 3 & 7 \\ 2 & -6 \end{bmatrix} * \begin{bmatrix} 6 & 7 \\ 2 & -3 \end{bmatrix}$$

$$= \frac{1}{32}\begin{bmatrix} 18+14 & 21-21 \\ & \end{bmatrix} \checkmark$$

Bonus Example:



The Matrix of a Reflection Across a Plane in $\mathbb{R}^3$

$$\boxed{refl_{\Pi} \circ refl_{\Pi} = Id_{\mathbb{R}^3}}$$

Let Π be the plane in $\mathbb{R}^3$ with Cartesian equation:

$$[refl_{\Pi}]^2 = I_3$$

$$3x - 5y + 2z = 0.$$

We found in the last Example of Section 2.2 that:

$$[refl_{\Pi}]^{-1} = [refl_{\Pi}]$$

$$[refl_{\Pi}] = \begin{bmatrix} \frac{10}{19} & \frac{15}{19} & -\frac{6}{19} \\ \frac{15}{19} & -\frac{6}{19} & \frac{10}{19} \\ -\frac{6}{19} & \frac{10}{19} & \frac{15}{19} \end{bmatrix}$$