$\boxed{2.7}$

$\boxed{\text{Thm}}$ $A_{n\times n}$ is invertible. $\boxed{\text{iff}}$

$$(A \mid I_n) \xrightarrow{GJR} (I_n \mid B)$$

then $\boxed{B = A^{-1}}$

Pf

($\Rightarrow$) A is invertible. So A^{-1} exists. WTS: $B = A^{-1}$.

$$B = [\vec{x}_1 \mid \vec{x}_2 \mid \cdots \mid \vec{x}_n].$$

GJR: $\underbrace{A\vec{x}_j = \vec{e}_j}_{(*)} \xleftrightarrow{GJR} (A \mid \vec{e}_j) \to (I_n \mid \vec{x}_j)$

using previous thm
$RREF(A) = I_n$

Thus:

$$(A \mid \vec{e}_1 \mid \vec{e}_2 \mid \cdots \mid \vec{e}_n) \xrightarrow{GJR} (I_n \mid \vec{x}_1 \mid \vec{x}_2 \mid \cdots \mid \vec{x}_n)$$

$$\parallel$$

$$(A \mid I_n) \longrightarrow (I_n \mid B)$$

so (*): $A * B = I_n$. So since A^{-1} exists,

$$A^{-1} * (A * B) = A^{-1} * I_n$$

$\underbrace{(A^{-1} * A)} * B = A^{-1}$ (Associativity Prop MM & Prop of I_n)

$I_n * B = A^{-1}$ (since A^{-1} is inverse of A)

$\boxed{B = A^{-1}}$ (prop of I_n).

($\Leftarrow$) Assume that $(A \mid I_n) \xrightarrow{GJR} (I_n \mid B)$

& $B = A^{-1}$. Nothing to prove!

$[I_n \mid B] \Longleftrightarrow A\vec{x}_j = \vec{e}_j \Longleftrightarrow A * B = I_n$

so B is inverse of A. $\square$

Example

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & 2 & 1 \end{bmatrix}$$

a) Find A^{-1} using GJR, write all EROs.

b) Express A as a product of elementary matrices.

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1 a) start: $(A \mid I_3)$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\textcircled{2}\ R_3 - 2R_1 \to R_3]{\textcircled{1}\ R_2 - R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & -4 & -1 & -2 & 0 & 1 \end{array}\right]$$

(3) $-R_2 \to R_2$

(4) $4R_2 + R_3 \to R_3$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 2 & -4 & 1 \end{array}\right]$$

(5) $-R_3 \to R_3$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 & 4 & -1 \end{array}\right]$$

(6) $R_1 - R_3 \to R_1$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 3 & -4 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 & 4 & -1 \end{array}\right]$$

(7) $R_1 - 3R_2 \to R_1$

$$\longrightarrow \left[\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I_3} & \underbrace{\begin{matrix} 0 & -1 & 1 \\ 1 & -1 & 0 \\ -2 & 4 & -1 \end{matrix}}_{A^{-1}} \end{array}\right]$$

(1) $R_2 - R_1 \to R_2$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(2) $R_3 - 2R_1 \to R_3$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

(3) $-R_2 \to R_2$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(4) $4R_2 + R_3 \to R_3$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \qquad E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}$$

(5) $-R_3 \to R_3$

$$E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad E_5^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(6) $R_1 - R_3 \to R_1$

$$E_6 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_6^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(7) $R_1 - 3R_2 \to R_1$

$$E_7 = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_7^{-1} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

so

$$\boxed{A = (E_1^{-1} * E_2^{-1}) * (E_3^{-1} * E_4^{-1}) * (E_5^{-1} * E_6^{-1}) * E_7^{-1}}$$

$$E_1^{-1} * E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \checkmark$$

$$E_3^{-1} * E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \checkmark$$

$$E_5^{-1} * E_6^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \checkmark$$

$$A \stackrel{?}{=} \left(\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \right) \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

= it works.... tired.... ~

Nice Bonus

Thm $\boxed{A^{-1} = E_k * E_{k-1} * \cdots * E_2 * E_1}$

Pf $A = E_1^{-1} * E_2^{-1} * \cdots * E_k^{-1}$

& E_i's are invertible (& E_i^{-1})

$$A * E_k = (E_1^{-1} * E_2^{-1} * \cdots * E_k^{-1}) * E_k$$

$$= (E_1^{-1} * E_2^{-1} * \cdots * E_{k-1}^{-1}) * (\underbrace{E_k^{-1} * E_k}_{I_n})$$

$$= (E_1^{-1} * E_2^{-1} * \cdots * E_{k-1}^{-1}) * I_n$$

$$= E_1^{-1} * \cdots * E_{k-1}^{-1}.$$

So multiply by E_{k-1}:

$$A * E_k * E_{k-1} = E_1^{-1} * \cdots * E_{k-2}^{-1} * \overbrace{(E_{k-1}^{-1} * E_{k-1})}^{I_n}$$

$$= E_1^{-1} * \cdots * E_{k-2}^{-1}$$

Continue in this way:

$$A * (E_k * E_{k-1} * \cdots * E_2 * E_1) = I_n$$

Since A^{-1} is unique,

$$A^{-1} = E_k * E_{k-1} * \cdots * E_1 .$$

$\square$

$\boxed{\text{Thm}}$ A is invertible $\boxed{\text{iff}}$ A is expressible as product of elementary matrices:

$$\boxed{A = E_1^{-1} * E_2^{-1} * \cdots * E_k^{-1}}$$

where $E_1, \ldots, E_k$ are the corresponding elementary matrics in the GJRA.

Pf $(\Rightarrow)$ Assume A is invertible. Then $RREF(A) = I_n$

so $E_1, E_2, \ldots, E_k$ exist.

$$\boxed{E_1 * A}$$

this results in A with the first ERO applied.

Then:

$$E_k * \cdots * E_2 * E_1 * A = I_n.$$

(Note $A^{-1} = E_k * \cdots * E_1$ by Uniqueness of Inv)

Since Elementary matrics are invertible: $E_1^{-1}, \ldots, E_k^{-1}$ exist.

so:

$$E_k^{-1} * (E_k * \cdots * E_1 * A) = E_k^{-1} * I_n$$

$$\underbrace{(E_k^{-1} * E_k)}_{I_n} * (E_{k-1} * \cdots * E_1 * A) = E_k^{-1}$$ (by Ass Prop of MM & Prop of I_n)

$$I_n * (\quad " \quad) = E_k^{-1}$$

$$E_{k-1} * E_{k-2} * \cdots * E_1 * A = E_k^{-1}$$

Next follow a similar proof.

$$\underbrace{E_{k-1}^{-1} * (E_{k-1}} * E_{k-2} * \cdots * E_1 * A) = E_{k-1}^{-1} * E_k^{-1}$$

$$E_{k-2} * \cdots * E_1 * A = E_{k-1}^{-1} * E_k^{-1}$$

$\cdots\cdots$

$$A = E_1^{-1} * E_2^{-1} * \cdots * E_{k-1}^{-1} * E_k^{-1}.$$

($\Leftarrow$) exercise.

$\square$