2.7 Finding the Inverse of a Matrix

Use GJR to find A^{-1}

Goal: to be able to construct the matrix of the inverse of an invertible linear operator, and at the same time, to find the inverse of an invertible square matrix which is 3×3 or bigger, when it is possible to do so.

Multiplicative Properties of Elementary Matrices

E elementary matrix = do 1 ERO to I_n.

Theorem: If $\boxed{E \text{ is an elementary } n \times n \text{ matrix}}$ and A is any $n \times m$ matrix, then the $\boxed{\textbf{\textit{matrix product } } EA}$ can be computed by simply performing the ***same elementary row operation*** on A that was used to produce E from I_n.

An elementary matrix ***encodes*** the elementary row operation that produced it.

Old fashioned

↳ but important conceptually:

b/c easy to "talk" to a computer

w/ matrices (ie #s)

& not easy to "talk"

Example: Suppose that

$$A = \begin{bmatrix} 5 & 7 & -2 & 3 \\ 4 & 1 & 8 & -5 \\ 2 & -3 & 9 & 6 \end{bmatrix}$$

and:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ERO

$3 * R_2 \to R_2$

$$E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

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$5 * R_2 + R_3 \to R_3$

Then:

$$E_1A = \underset{3\times 3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{3\times 4}{\begin{bmatrix} 5 & 7 & -2 & 3 \\ 4 & 1 & 8 & -5 \\ 2 & -3 & 9 & 6 \end{bmatrix}}$$

$$= \underset{3\times 4}{\begin{bmatrix} 5 & 7 & -2 & 3 \\ 12 & 3 & 24 & -15 \\ & & & \end{bmatrix}},$$

$$E_2A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 7 & -2 & 3 \\ 4 & 1 & 8 & -5 \\ 2 & -3 & 9 & 6 \end{bmatrix}$$

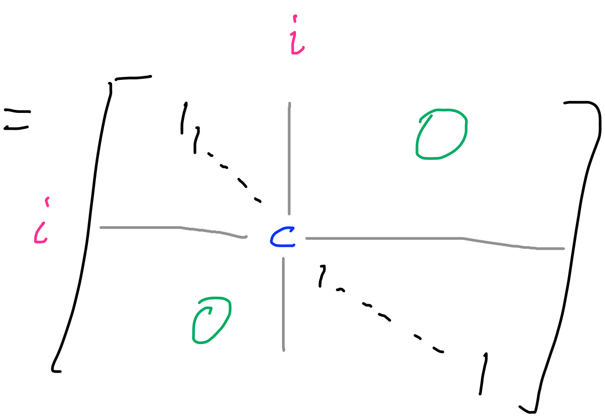
$$= \underset{3\times 4}{\begin{bmatrix} 2 & -3 & 9 & 6 \\ & & & \\ 5 & 7 & -2 & 3 \end{bmatrix}}, \quad \text{and}$$

$$E_3A = \begin{bmatrix}1&0&0\\0&1&0\\0&5&1\end{bmatrix}\begin{bmatrix}5&7&-2&3\\4&1&8&-5\\2&-3&9&6\end{bmatrix}$$

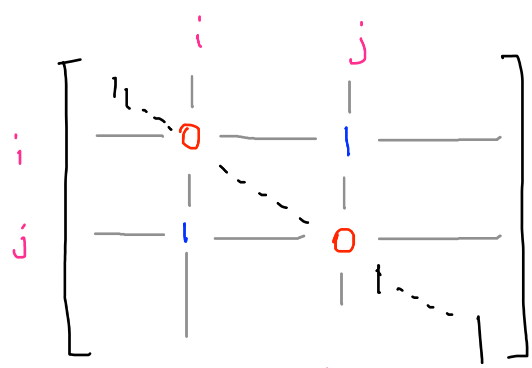
$$= \begin{bmatrix}5&7&-2&3\\4&1&8&-5\\22&2&49&-19\end{bmatrix}_{3\times 4}$$

EROs

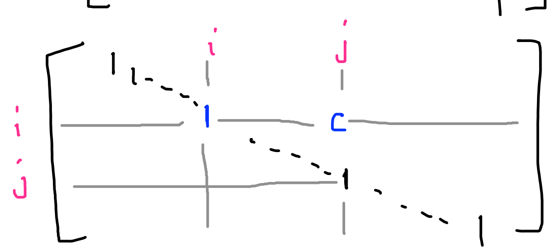
(1) $cR_i \to R_i$

$E =$ 

(2) $R_i \leftrightarrow R_j$

$E =$ 

(3) $R_i + cR_j \to R_i$

$E =$ 

Theorem: Elementary matrices are ***invertible***, and the inverse of an elementary matrix is another elementary matrix of exactly the ***same type***.

Pf ✓ □

Examples:

$$\text{For } E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\text{For } E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

$$\text{For } E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}, \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}.$$

$5R_2 + R_3 \to R_3$

A Preliminary Test for Invertibility

Important: "Quick" check of invertibility

Theorem: Let A be an $n \times n$ matrix. Then $\boxed{A \text{ is invertible}}$ ***if and only if*** the $\boxed{\text{rref of } A \text{ is } I_n.}$

A is invertible $\boxed{\text{iff}}$ $RREF(A) = I_n$.

Pf ($\Rightarrow$) A is invertible $\Rightarrow$ A^{-1} exists & T is 1-1 & onto

- In Ch1: saw $RREF(A) = I_n$ or get entire row of 0s.
  ↳ free variables

If there's a row of 0s in $RREF(A)$ then there's at least 1 free variable $\Rightarrow$ $NS(A)$ has at least one nonzero vector

$\dim(NS(A)) \geq 1$ so $\ker(T) \neq \{\vec{0}\}$ so T is not 1-1! So this contradiction & can't row of 0s

so $RREF(A) = I_n$.

($\Leftarrow$) Assume $RREF(A) = I_n$. $\Rightarrow$ $NS(A) = \{\vec{0}\}$

$\Rightarrow \ker(T)$ is 1-1. ✓

Next we show T is onto:

Since $RREF(A) = I_n$, $CS(A) = \text{Span}(\{\vec{v}_1, \dots, \vec{v}_n\})$

where v_i's are n LI $\Rightarrow CS(A) = \mathbb{R}^n$.

$\Rightarrow \text{range}(T) = \mathbb{R}^n$, so T is onto. ✓

so T is invertible $\Rightarrow$ A is invertible. $\square$

A Method to Find A^{-1}

... efficiently !!

Fundamental Result of LA

Theorem: Let A be an $n \times n$ matrix. If we construct the $n \times 2n$ augmented matrix:

$$\left[\, A \mid I_n \,\right],$$

then A is invertible *if and only if* the rref of this augmented matrix contains I_n in the first n columns. If this is the case, then A^{-1} will be found in the last n columns. In other words, the rref of $\left[\, A \mid I_n \,\right]$ is:

$$\left[\, I_n \mid A^{-1} \,\right]$$

$$(A \mid I_n) \xrightarrow[\text{RREF}]{\text{GJR}} (I_n \mid A^{-1})$$

Key Idea: there are only two possibilities for the rref of a square matrix.

Factoring Invertible Matrices

Theorem: An $n \times n$ matrix A is invertible *if and only if* it can be expressed as a product of elementary matrices. If this is the case, then more precisely, we can factor A as:

$$\boxed{A = E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1},}$$

(Not unique!)

where $E_1, E_2, \ldots, E_k$ are the elementary matrices corresponding to a choice of elementary row operations we used in the Gauss-Jordan Algorithm to transform A into I_n.

Note: The factorization of A into elementary matrices is ***not unique***, since a different choice of elementary row operations will result in a different factorization.

Solving Invertible Square Equations

Important victory lap :-) The point is we can solve "matrix equations" in notation that is very similar to regular algebra (in 1 variable).

Theorem: If A is an invertible $n \times n$ matrix, then the system:

Matrix EQ: $$\boxed{A\vec{x} = \vec{b}}$$

has exactly one solution for any $n \times 1$ matrix $\vec{b}$, namely:

$$\boxed{\vec{x} = A^{-1}\vec{b}.}$$

More generally, if C is any $n \times m$ matrix, then the matrix equation:

$$\boxed{AB = C}$$

has exactly one solution for the $n \times m$ matrix B, namely:

$$\boxed{B = A^{-1}C}$$

Pf A is invertible so A^{-1} exists.

$$\boxed{A\vec{x} = \vec{b}}$$

$$\underbrace{A^{-1}_{n\times n} * \underbrace{(A_{n\times n}\vec{x}_{n\times 1})}_{n\times 1}}_{n\times 1} = \underbrace{A^{-1}_{n\times n} * \vec{b}_{n\times 1}}_{n\times 1}$$

$$(\underbrace{A^{-1} * A}_{I_n}) * \vec{x} = A^{-1} * \vec{b}$$

$$I_n * \vec{x} = A^{-1} * \vec{b}$$

$$\boxed{\vec{x} = A^{-1} * \vec{b}}$$

Pf $\underbrace{A_{n\times n} * X_{n\times m}} = C_{n\times m}$

Solve for $X_{n\times m}$

$$A^{-1} * (A * X) = A^{-1} * C$$

$$(\underbrace{A^{-1} * A}_{I_n}) * X = A^{-1} * C$$

$$\boxed{X = A^{-1} * C}$$

□

Good test Q ;-)