2.8

Thm One Sided Inverse is Sufficient

A is invertible $\boxed{\text{iff}}$ $\exists\, B_{n\times n}$ st

$A*B = I_n$ OR $B*A = I_n$

Pf ($\Longrightarrow$) ✓

($\Longleftarrow$) Assume $\boxed{A*B = I_n}$.

Assume $\exists\, B$,

NTS A is invertible:

(1) T associated to A is 1-1

(2) T onto.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be LO associated to A:

$$T(\vec{v}) = A\vec{v}.$$

Let $S : \mathbb{R}^n \to \mathbb{R}^n$ be LO associate to B:

$$S(\vec{v}) = B\vec{v}$$

Then $T \circ S = Id_{\mathbb{R}^n}$:

Because: $\forall \vec{v} \in \mathbb{R}^n$:

$$
\begin{aligned}
(T \circ S)(\vec{v}) &= T(S(\vec{v})) \\
&= T(B\vec{v}) \\
&= A(B\vec{v}) \\
&= (A * B)\vec{v} \\
&= I_n \vec{v} \qquad \text{(by assumption)} \\
&= \vec{v}.
\end{aligned}
$$

(Note: really interesting argument:

$T \circ S = Id_{\mathbb{R}^n} \Longrightarrow T$ is onto & 1-1)

- T is onto: let $\vec{w} \in \mathbb{R}^n$ NTS: $\exists \vec{v} \in \mathbb{R}^n$: $T(\vec{v}) = \vec{w}$.

We know:

$$
\begin{aligned}
\vec{w} &= (T \circ S)(\vec{w}) \qquad \text{(b/c } T \circ S = Id_{\mathbb{R}^n}\text{)} \\
&= T(\underbrace{S(\vec{w})})
\end{aligned}
$$

let $\vec{v} = S(\vec{w})$ so $\vec{w} = T(\vec{v})$. So T onto ✓

- <u>T is 1-1:</u>

  By Dimension Theorem;

  $$\text{rank}(T) + \text{nullity}(T) = n$$

  Since T is onto, $\text{rank}(T) = n$ (b/c $\text{Range}(T) = \mathbb{R}^n$)

  Thus, $\text{nullity}(T) = n - \text{rank}(T) = n - n = 0$.

  This says $\dim(\ker(T)) = 0 \Rightarrow \ker(T) = \{\vec{0}\}$.

  Thus, T is 1-1.

- Since T is onto & 1-1, T is invertible

  hence A is invertible ($A = [T]$).

- Finally, we can now prove that $B = A^{-1}$:

$$B = I_n * B$$

$$= (A^{-1} * A) * B \qquad \text{(since } A \text{ is invertible we know } \exists A^{-1}\text{)}$$

$$= A^{-1} * (\underbrace{A * B}) \qquad \text{(Ass.)}$$

$$= A^{-1} * I_n \qquad \text{(by assumption)}$$

$$= A^{-1} \qquad \text{(Prop of } I_n\text{)}$$

so $B = A^{-1}$.

So get for free $B * A = I_n$.

$\square$

Thm If T_1 & T_2 LOs on $\mathbb{R}^n$

Then $T_2 \circ T_1$ (& $T_1 \circ T_2$) is invertible

&

$$\boxed{[T_2 \circ T_1]^{-1} = [T_1]^{-1} * [T_2]^{-1}}$$

Pf We already know:

$$[T_2 \circ T_1] = [T_2] * [T_1].$$

So if the inverse exists, call it B, then it must satisfy:

$$\underline{B * [T_2 \circ T_1] = I_n} \quad \& \quad \underline{[T_2 \circ T_1] * B = I_n}$$

But T_1 & T_2 are invertible, so $[T_1]^{-1}$, $[T_2]^{-1}$ exist.

Observe then:

$$[T_1]^{-1} * [T_2]^{-1} * [T_2 \circ T_1]$$

$$= [T_1]^{-1} * \Big(\underbrace{[T_2]^{-1} * [T_2]}_{I_n}\Big) * [T_1] \quad \text{(By Big Result on compositions) \& Ass. prop}$$

$$= [T_1]^{-1} * I_n * [T_1]$$

$$= \underbrace{[T_1]^{-1} * [T_1]}_{I_n}$$

$$= I_n.$$

So $[T_2 \circ T_1]^{-1} = [T_1]^{-1} * [T_2]^{-1}$ by One-Sided Inverses is Enough Theorem. Consequently, $T_2 \circ T_1$ is invertible.

$\square$