$(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \cdots\cdots \Leftrightarrow (22)$

2.8 Consequences of Invertibility

"TFAE"

Memorize!

Theorem — The Really Big Theorem on Invertibility:

The following conditions are equivalent for a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$, with standard matrix $[T] = A$:

1. T is an invertible operator.
2. A is an invertible matrix.
3. The rref of A is I_n.
4. A is the product of elementary matrices.
5. T is one-to-one.
6. $ker(T) = nullspace(A) = \left\{\vec{0}_n\right\}$.
7. $nullity(T) = nullity(A) = 0$.
8. T is onto.
9. $range(T) = \mathbb{R}^n$.
10. $rank(T) = n$.

Dim Th $\quad T : \mathbb{R}^n \to \mathbb{R}^m$

$rank(T) + nullity(T) = n$

$dim(range(T)) + dim(ker(T)) = n$

$\rightarrow$ range(T)

11. $colspace(A) = \mathbb{R}^n$.

12. The columns of A are linearly independent.

13. The columns of A Span $\mathbb{R}^n$.

14. The columns of A form a basis for $\mathbb{R}^n$.

15. $rowspace(A) = \mathbb{R}^n$.

16. The rows of A are linearly independent.

17. The rows of A Span $\mathbb{R}^n$.

18. The rows of A form a basis for $\mathbb{R}^n$.

19. The homogeneous equation $A\vec{x} = \vec{0}_n$ has only the trivial solution.

20. ***For every*** $n \times 1$ matrix $\vec{b}$, the system $A\vec{x} = \vec{b}$ is ***consistent***.

21. ***For every*** $n \times 1$ matrix $\vec{b}$, the system $A\vec{x} = \vec{b}$ has ***exactly one solution***.

22. ***There exists*** an $n \times 1$ matrix $\vec{b}$, such that the system $A\vec{x} = \vec{b}$ has ***exactly one solution***.

Read and Study this proof from the book! Some of the directions are big results and some of them are definitions. Some of them are easy directions. I showed a few of them in class: (7) <=> (10)

One Sided Inverses

Given A, we must find B so that:

$$AB = I_n \boxed{\textbf{\textit{and}}} BA = I_n$$

If B only satisfies the first equation, we call B a “right” inverse for A.

If B only satisfies the second equation we call B a “left” inverse for A.

Luckily, there’s no need for this nonsense:

Theorem: An $n \times n$ matrix A is invertible if and only if we can find an $n \times n$ matrix B such that $AB = I_n$ ***or*** $BA = I_n$. Thus, a "right" inverse is also a "left" inverse, and vice versa.

This is subtle and deep result. It has a nice proof too! See the extra notes.

Proof: think of BA as a matrix representing the composition of two operators.

The Inverse of a Composition and Matrix Product

Theorem: If $T_1, T_2 : \mathbb{R}^n \to \mathbb{R}^n$ are both invertible operators, then $T_2 \circ T_1$ is also invertible, and furthermore:

$$\boxed{[T_2 \circ T_1]^{-1} = [T_1]^{-1}[T_2]^{-1}.}$$

Analogously, if A and B are invertible $n \times n$ matrices, then AB is also invertible, and furthermore:

$$(AB)^{-1} = B^{-1}A^{-1}.$$

`Another key result in Linear Algebra!`

The converse is also true!

`By the "Equivalence of LT & Mat Thm" we immediately get:`

Theorem: If $T_1, T_2 : \mathbb{R}^n \to \mathbb{R}^n$ are operators and the ***composition*** $T_2 \circ T_1$ is ***invertible***, then ***both*** T_2 and T_1 are also invertible. Analogously if A and B are two $n \times n$ matrices and the ***product*** AB is ***invertible***, then ***both*** A and B are invertible.