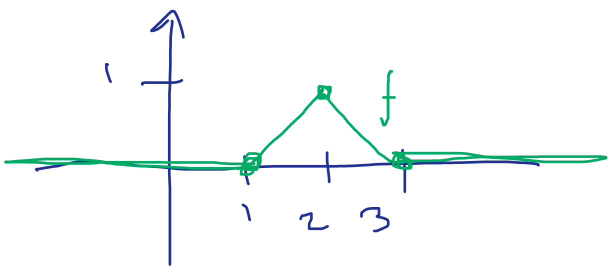
$\boxed{3.1}$

Examples

- **Function Spaces:**

$\mathcal{F}(\mathbb{R},\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \mid \text{function} \}$

$C(\mathbb{R},\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} / \text{continuous} \}$



$$f(x) = \begin{cases} 0, & x < 1, x > 3 \\ x-1, & 1 \le x \le 2 \\ -x+3, & 2 \le x \le 3 \end{cases}$$

$f \in C(\mathbb{R},\mathbb{R})$

- $I(\mathbb{R},\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \mid \text{integrable} \}$

$\hookrightarrow \int_{-\infty}^{\infty} f \, dx < \infty$

$f \in I(\mathbb{R},\mathbb{R})$

$$\int_{-\infty}^{\infty} f(x) dx < \infty$$

Recall: f is integrable if the integral exists (limit of Riemann sums exists) and is finite.

- $D(\mathbb{R},\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} / f \text{ is differentiable} \}$

$f'(x)$ exist $\forall x$

$f(x) = e^x \;,\; f'(x) = e^x$

these are all subspaces of $\mathcal{F}(\mathbb{R}, \mathbb{R})$

- $L^2([0,1]) = \{ f : [0,1] \to \mathbb{R} \mid \int_0^1 f^2(x)\,dx < \infty \}$

$f, g \in L^2([0,1])$ then $f+g$?

$$\int_0^1 (f+g)^2 dx \le \int_0^1 \Big(2\underbrace{\max(f,g)}_{}\Big)^2 dx$$

↳ this whichever function is bigger (why I put a 2 in front)

$$= 4\int_0^1 [\max(f,g)]^2 dx$$

$$\le 4\int_0^1 [f^2+g^2]\, dx$$ (b/c $\max(f,g)$ is one of them so $\max(f,g)^2 \le f^2+g^2$)

$$= 4\underbrace{\int_0^1 f^2 dx}_{<\infty} + 4\underbrace{\int_0^1 g^2 dx}_{<\infty} < \infty$$

- MORE Examples from Calculus

ℓ_∞ "ell infinity"

$\ell_\infty = \{$ sequences of real numbers & bounded $\}$

notation $\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \ldots\}$

$= \langle a_1, a_2, a_3, \ldots \rangle$

$$\langle a_n \rangle_{n=1}^{\infty} \in \ell_\infty$$

- Define $\oplus$:

$$\langle a_n \rangle_{n=1}^{\infty} \oplus \langle b_n \rangle_{n=1}^{\infty} = \langle a_1+b_1,\ a_2+b_2,\ \dots,\ a_n+b_n,\ \dots \rangle$$

- Define $\odot$:

$$r \odot \langle a_n \rangle_{n=1}^{\infty} = \langle ra_1,\ ra_2,\ \dots,\ ra_n,\ \dots \rangle$$

☆ $\vec{0}_{\ell_\infty} = \langle 0, 0, 0, \dots, 0, \dots \rangle$

Can check that all the VSAs hold! (Exercise).

☆ $-\langle a_n \rangle_{n=1}^{\infty} = \langle -a_1, -a_2, \dots, -a_n, \dots \rangle$

<u>Ex</u> $\langle 1, 1, 1, \dots \rangle \in \ell_\infty$

$\langle 1, 2, 3, 4, \dots \rangle \notin \ell_\infty$

Can check that all the VSAs hold! (Exercise).

- $$\ell_1 = \left\{ \langle a_n \rangle_{n=1}^{\infty} \;\middle|\; \sum_{n=1}^{\infty} |a_n| < \infty \right\}$$

$= \{$ absolutely convergent sequences $\}$.

- $$\ell_2 = \left\{ \langle a_n \rangle_{n=1}^{\infty} \;\middle|\; \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}$$

Can check that all the VSAs hold! (Exercise).

Thm Uniqueness of Zero Vector.

If there's another $\vec{0}_2$ with the properties: $\forall \vec{v} \in V$

(*) $\boxed{\vec{0}_2 \oplus \vec{v} = \vec{v}}$ & $\boxed{\vec{v} \oplus \vec{0}_2 = \vec{v}}$

then

$$\vec{0}_V = \vec{0}_2 .$$

Pf By assumption $\vec{0}_2 \in V$, so VSA 5:

$$\vec{0}_V \oplus \vec{0}_2 = \vec{0}_2 .$$

Also, $\vec{0}_V \in V$ so by assumption (*):

$$\vec{0}_2 \oplus \vec{0}_V = \vec{0}_V .$$

Then

$$\vec{0}_2 = \vec{0}_V \oplus \vec{0}_2$$

$$= \vec{0}_2 \oplus \vec{0}_V \qquad (\text{VSA } 3)$$

$$= \vec{0}_V .$$

$\square$