3.1 Axioms for a $\boxed{\text{Vector Space}}$

Note: Memorize all of these! I prefer the version I wrote on the white-board but these are ok too.

Definition — The Axioms of an (Abstract) Vector Space:

A vector space $(V, \oplus, \odot)$ is a $\boxed{\text{non-empty set } V,}$ together with two operations:

1. $\boxed{\oplus \quad \text{(vector addition),}}$ and

2. $\boxed{\odot \quad \text{(scalar multiplication),}}$

such that: for all $\vec{u}$, $\vec{v}$ and $\vec{w} \in V$ and all r, $s \in \mathbb{R}$, $(V, \oplus, \odot)$ satisfies the following ten properties:

1. *The Closure Property of Vector Addition:*

$$\vec{u} \oplus \vec{v} \in V$$

$\forall \vec{u}, \vec{v} \in V$

2. *The Closure Property of Scalar Multiplication:*

$$r \odot \vec{u} \in V$$

$\forall r \in \mathbb{R}$

3. *The Commutative Property of Vector Addition:*

$$\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$$

4. *The Associative Property of Vector Addition:*

$$(\vec{u} \oplus \vec{v}) \oplus \vec{w} = \vec{u} \oplus (\vec{v} \oplus \vec{w})$$

5. *The Existence of a Zero Vector:*

There exists $\vec{0}_V \in V$, such that: $\vec{0}_V \oplus \vec{v} = \vec{v} = \vec{v} \oplus \vec{0}_V$

6. *The Existence of Additive Inverses:*

There exists $-\vec{v} \in V$ such that: $\vec{v} \oplus (-\vec{v}) = \vec{0}_V = (-\vec{v}) \oplus \vec{v}$

7. ***The Distributive Property of Ordinary Addition over Scalar Multiplication:***

$$(r+s)\odot\vec{v} = (r\odot\vec{v})\oplus(s\odot\vec{v})$$ $(\forall r, s \in \mathbb{R})$

8. ***The Distributive Property of Vector Addition over Scalar Multiplication:***

$$r\odot(\vec{u}\oplus\vec{v}) = (r\odot\vec{u})\oplus(r\odot\vec{v})$$

9. ***The Associative Property of Scalar Multiplication:***

$$r\odot(s\odot\vec{v}) = s\odot(r\odot\vec{v}) = (rs)\odot\vec{v}$$

10. ***The Unitary Property of Scalar Multiplication:***

$$\boxed{1\odot\vec{v} = \vec{v}}$$

Note Memorize these!

↳ will be on test!

We need ***three objects,*** that is, three pieces of ***information*** to define a vector space:

(1) a non-empty ***set*** V,

(***what*** are the vectors)

(2) a rule for ***vector addition*** $\oplus$ that tells us ***how to add*** two vectors to get another vector, and

(3) a rule for ***scalar multiplication*** $\odot$ that tells us ***how to multiply*** a real number with a vector to get another vector.

Polynomial Spaces

$\mathbb{P}^n = \{$ all $\underline{\text{n-degree}}$ polynomials $\}$

$$\mathbb{P}^n = \{p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mid a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}\}$$

Example: $\mathbb{P}^2$

$$\underline{p(x) = 3 - 5x + 7x^2} \quad \text{and}$$

$$\underline{q(x) = 4 - 3x^2} \in \mathbb{P}^2$$

$$\underline{p(x) \oplus q(x)} = (3 - 5x + 7x^2) + (4 - 3x^2)$$
$$= 7 - 5x + 4x^2, \quad \text{and}$$

$$\underline{3 \odot p(x)} = 3(3 - 5x + 7x^2)$$
$$= 9 - 15x + 21x^2$$

$$\boxed{\vec{0}_{\mathbb{P}^n} = z(x)} = 0 + 0x + \cdots + 0x^n$$

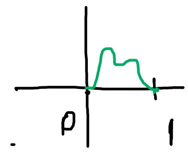
$$-p(x) = -a_0 - a_1 x - a_2 x^2 - \cdots - a_n x^n$$

all VSA are def true!

$\mathcal{F}([0,1], \mathbb{R})$

Functions Spaces

I = interval in $\mathbb{R}$ or all $\mathbb{R}$.



$$F(I) = \{f(x) \mid f(a) \text{ is defined for all } a \in I\}$$

$f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid \text{function}\}$

$$\boxed{\begin{aligned}(f+g)(x) &= f(x) + g(x), \\ (kf)(x) &= k \cdot f(x)\end{aligned}} \quad \text{and}$$

$f \oplus g = f(x) + g(x) \qquad k \odot f = k f(x)$

The zero vector is simply the function $z(x)$ which outputs the value 0 for all $a \in I$.

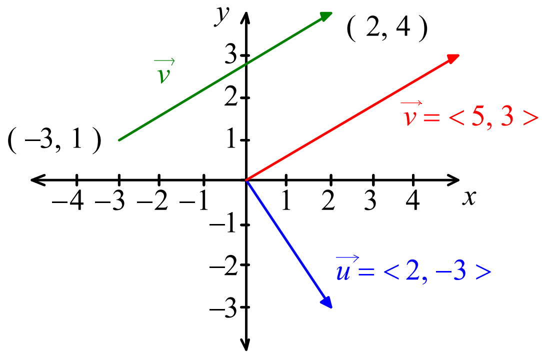
$\boxed{\vec{0}_{\mathcal{F}(\mathbb{R},\mathbb{R})} = z(x)}$, $\boxed{z(x) = 0.\ \forall x \in \mathbb{R}}$

The negative of a function is simply defined by the function which outputs as its value of $-f(a)$, with input $x = a$.

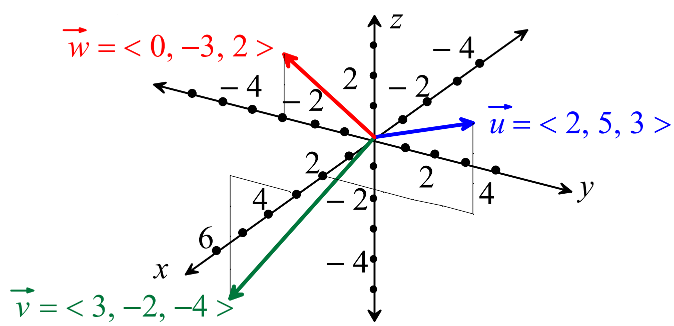
$-f$: additive inverse of f

$(-f)(x) = -f(x)$

How Can We Visualize Vectors?



Two Vectors, $\vec{u}$ and $\vec{v}$, in $\mathbb{R}^2$



Three Vectors, $\vec{u}$, $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$

Key point: polynomials are definitely not "straight". But they still satisfy the Vector Space Axioms! Thus, when we think of Vector Spaces are being "linear" we are only refering to the "flatness" or "straightness" in the context of Euclidean Spaces R^n!!!

$\mathbb{R}^4$???

$\mathbb{P}^3$???

$F(\mathbb{R})$???

Polynomial Spaces and, more generally, the Function Spaces are not "flat" or "straight" when we view them as graphs. The "linear structure", then, refers to the META properties: you can add two functions and create a new function; you can multiply a function by a scalar and create a new function.

$m \times n$ *Matrices*

$\mathbb{M}_{m\times n}(\mathbb{R}) = \{\text{all } m\times n \text{ matrices}\}$

Go back and read section 2.4. The first theorem verifies all the vector space axioms for Matrices!

$$Mat(m,n) = \{A \mid A \text{ is an } m \times n \text{ matrix}\}$$

$A \oplus B = (a_{ij})_{m\times n} \oplus (b_{ij})_{m\times n} = (a_{ij} + b_{ij})_{m\times n}$

$r \odot A = r \odot (a_{ij})_{m\times n} = (r a_{ij})_{m\times n}$

The Smallest Example

$$\boxed{V = \{\vec{0}_V\}}$$

Addition? Scalar Multiplication?

By $V \neq \emptyset$ so $\exists\, \vec{v} \in V$.

By VSA5: $\vec{0}_V \in V$.

Define: $\vec{0}_V \oplus \vec{0}_V = \vec{0}_V$

Define $-\vec{0}_V = \vec{0}_V$ so $\vec{0}_V \oplus -\vec{0}_V = \vec{0}_V \oplus \vec{0}_V = \vec{0}_V$.

We're Not in Kansas Anymore

$$\mathbb{R}^+ = \left\{\vec{x} | x \in \mathbb{R}, \text{ and } x > 0\right\},$$

$$\vec{x} \oplus \vec{y} = \overrightarrow{xy} \qquad \text{(ordinary multiplication)}$$

$$r \odot \vec{x} = \overrightarrow{x^r} \qquad \text{(ordinary exponentiation)}$$

$$= \overrightarrow{e^{r\ln(x)}}$$

Identity element:

$$\vec{z} \oplus \vec{y} = \vec{y}$$

$$\vec{z} = ???$$

$$\overrightarrow{0}_{\mathbb{R}^+} = ???$$

Inverses:

$$\vec{x} \oplus \vec{y} = \overrightarrow{0}_{\mathbb{R}^+}$$

$$\vec{y} = ???$$

Last four Axioms:

$$(r + s) \odot \vec{x} = ???$$

$$r \odot (\vec{x} \oplus \vec{y}) = ???$$

$$(rs) \odot \vec{x} = ???$$

$$1 \odot \vec{x} = ???$$

Additional Properties of Vector Spaces

Theorem — ***The Uniqueness of the Zero Vector:***

The ***zero vector*** $\vec{0}_V$ of any vector space $(V, \oplus, \odot)$ is ***unique***. This means that if $\vec{z} \in V$ is another vector that satisfies: $\vec{z} \oplus \vec{v} = \vec{v}$ for ***all*** $\vec{v} \in V$, then we must have: $\vec{z} = \vec{0}_V$.

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Classic: one of the first proofs everyone learns when dealing
with abstract vector spaces. Seems a bit silly, but it's
important to learn to prove things from axioms.
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Theorem — ***The Uniqueness of Additive Inverses:***

The ***additive inverse*** $-\vec{v}$ of any vector $\vec{v} \in V$ in a vector space $(V, \oplus, \odot)$ is ***unique***. This means that if $\vec{n} \in V$ is another vector that satisfies: $\vec{v} \oplus \vec{n} = \vec{0}_V$, then we must have: $\vec{n} = -\vec{v}$.
As a further consequence: $-\vec{v} = -1 \odot \vec{v}$.

↳ good exercise & test Q!

Theorem — The Multiplicative Properties of Zeroes:

Let $(V, \oplus, \odot)$ be a vector space, with zero vector $\vec{0}_V$. Then we have the following properties:

1. ***The Multiplicative Property of the Scalar Zero:***

$$\boxed{0 \odot \vec{v} = \vec{0}_V} \text{ for all } \vec{v} \in V.$$

2. ***The Multiplicative Property of the Zero Vector:***

$$\boxed{r \odot \vec{0}_V = \vec{0}_V} \text{ for all } r \in \mathbb{R}.$$

3. ***The Zero-Factors Theorem:*** For all $\vec{v} \in V$ and $r \in \mathbb{R}$:

$$\boxed{r \odot \vec{v} = \vec{0}_V} \text{ *if and only if* } \left(\text{either } r = 0 \textbf{ or } \vec{v} = \vec{0}_V.\right)$$

Pf 1: Let $\vec{v} \in V$ be arbitrary.

By Properties of $\mathbb{R}$: $0 + 0 = 0$.

Then $0 \odot \vec{v} = (0+0) \odot \vec{v}$

(*) $0 \odot \vec{v} = 0 \odot \vec{v} \oplus 0 \odot \vec{v}$ (VSA7)

By VSA2, $0 \odot \vec{v} \in V$ so by VSA6, $\exists -(0 \odot \vec{v}) \in V$

$$0 \odot \vec{v} \oplus [-(0 \odot \vec{v})] = \vec{0}_V.$$

VSA4

By substitution principle on (*):

$$\underbrace{0 \odot \vec{v} \oplus [-(0 \odot \vec{v})]} = (0 \odot \vec{v} + 0 \odot \vec{v}) \oplus [-(0 \odot \vec{v})]$$

$\vec{0}_V = 0 \odot \vec{v} \oplus \big(\underbrace{0 \odot \vec{v} \oplus -(0 \odot \vec{v})} \big)$

$\boxed{\vec{0}_V} = 0 \odot \vec{v} \oplus \vec{0}_V = \boxed{0 \odot \vec{v}}$ (VSA5) $\square$

Definition — Axiom for Parallel Vectors:

Let $(V, \oplus, \odot)$ be a vector space, and let $\vec{u}, \vec{v} \in V$. We say that $\vec{u}$ and $\vec{v}$ are *parallel to each other* if there exists either $a \in \mathbb{R}$ ***or*** $b \in \mathbb{R}$ such that:

$$\boxed{\vec{u} = a \odot \vec{v}} \;\; \textbf{\textit{or}} \;\; \boxed{\vec{v} = b \odot \vec{u}.}$$

Consequently, this means that $\vec{\mathbf{0}}_V$ is parallel to *all* vectors $\vec{v} \in V$, since $\vec{\mathbf{0}}_V = 0 \odot \vec{v}$.

no picture! def.

We're applying definitions to abstract objects based on our intuition of R^n (ok, really R^2 and R^3).

Things Don't Always Work Out

Example: Suppose $V = Mat(2,3)$, with vector addition defined as matrix addition, as before.

However, we will define scalar multiplication by:

$$r \odot A = r \odot \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}$$

$$r \odot A = \begin{bmatrix} ra_{1,1} & ra_{1,2} & ra_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}$$

Do the Distributive Properties still hold? Study VSA 7

$$(r+s) \odot A = \begin{bmatrix} (r+s)a_{11} & (r+s)a_{12} & (r+s)a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$s \odot A = \begin{bmatrix} sa_{11} & sa_{12} & sa_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$\neq$$

$$(r \odot A) \oplus (s \odot A) = \begin{bmatrix} ra_{11}+sa_{11} & ra_{12}+sa_{12} & ra_{13}+sa_{13} \\ 2a_{21} & 2a_{22} & 2a_{23} \end{bmatrix}$$

Example: Suppose we let $V = \mathbb{R}^2$, but with addition defined by:

$$\langle x_1, y_1 \rangle \oplus \langle x_2, y_2 \rangle = \langle 2x_1 + 2x_2, 2y_1 + 2y_2 \rangle.$$

Scalar multiplication: same as before.

Is there a zero vector? $\vec{0}_V = \langle a, b \rangle$

Does a vector have a negative?

$$\langle x, y \rangle \oplus \langle a, b \rangle = \langle x, y \rangle \quad \forall \langle x, y \rangle$$

$$\langle 2x + 2a, 2y + 2b \rangle = \langle x, y \rangle$$

$$\begin{cases} 2x + 2a = x & x = 0 \longrightarrow a = 0 \\ 2y + 2b = y & y = 0 \longrightarrow b = 0 \end{cases}$$

$$x = 1 \longrightarrow a = -1/2$$
$$y = 1 \longrightarrow b = -1/2$$

Not good!