3.2 Linearity Properties for Finite Sets of Vectors

Important Concepts in Linear Algebra: Vector Space (check), Linear Combinations, Linear Dependence & Independence, Subspaces. We studied all of these properties for Euclidean Spaces R^n. Now we revist these concepts for Abstract Vector Spaces.

Linear Combinations and Spans of Finite Sets of Vectors

$S = \{\vec{v}_1, \dots, \vec{v}_n\} \subseteq V$

Definition: Let $\boxed{S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}}$ be a set of vectors from a vector space $(V, \oplus, \odot)$, and

let $r_1, r_2, \dots, r_n \in \mathbb{R}$. Then, a ***linear combination*** of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ with

coefficients $r_1, r_2, \dots, r_n$ is an expression of the form:

LC $$\boxed{(r_1 \odot \vec{v}_1) \oplus (r_2 \odot \vec{v}_2) \oplus \cdots \oplus (r_n \odot \vec{v}_n).} \in V$$

Similarly, the ***Span*** of the set of vectors $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is the set of ***all possible linear combinations*** of these vectors:

$$\begin{aligned} Span(S) &= Span(\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}) \\ &= \{(r_1 \odot \vec{v}_1) \oplus (r_2 \odot \vec{v}_2) \oplus \cdots \oplus (r_n \odot \vec{v}_n) \mid \\ &\quad r_1, r_2, \dots, r_n \in \mathbb{R}.\} \end{aligned}$$

Span(S) = { all LC of vectors in S }

Example: The vector space $\mathbb{P}^n$ consists of all polynomials of degree at most n.

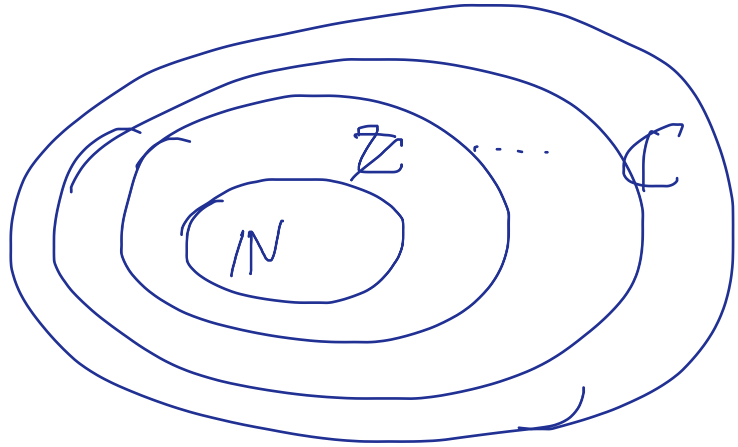
$$\mathbb{P}^n = \mathrm{Span}\left(\{1, x, x^2, \ldots\ldots, x^n\}\right)$$

why? $p(x) \in \mathbb{P}^n$ if

$$p(x) = c_0 1 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n$$

Remark $\dim \mathbb{P}^n = n+1$

Membership in A Span



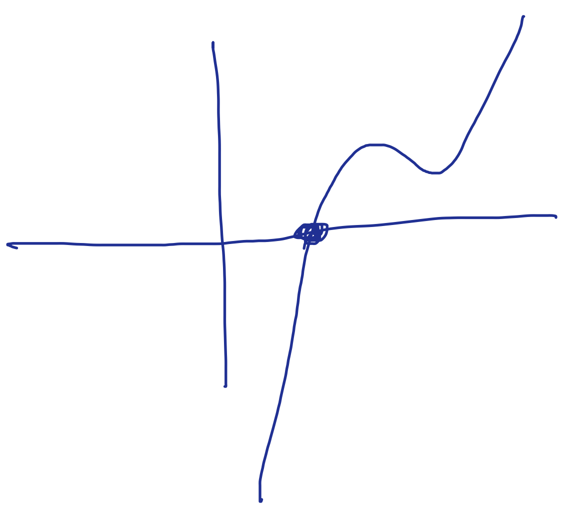
A useful theorem:

Theorem — The Fundamental Theorem of Algebra:

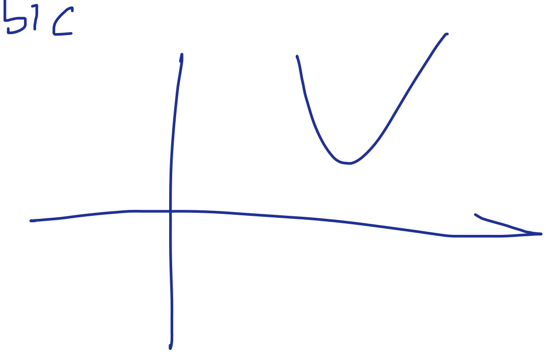
Every non-constant polynomial $p(x)$ (that is, of degree $n \geq 1$), with complex (or possibly real) coefficients, has exactly n complex roots, counting multiplicities.

$p(x) = 1 + x^2 \longrightarrow 2$ complex roots

$x = i, -i$



$p(x)$ cubic



Consequence:

Theorem* — *Equality of Polynomials: Suppose that:

- $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$ and
- $q(x) = d_0 + d_1 x + d_2 x^2 + \cdots + d_n x^n$.

Then, ***as functions***, $\boxed{p(x) = q(x)}$ (i.e. the graphs of the two functions are the same) ***if and only if*** $c_0 = d_0$, $c_1 = d_1$, …, $c_n = d_n$.

Note: We say that $p(x) = q(x)$ ***as functions*** if the values of the two functions agree for all real numbers $a \in \mathbb{R}$, that is:

$$p(a) = q(a) \text{ for all } a \in \mathbb{R}.$$

In other words, they have the same ***graph***.

Example: Consider the set S of polynomials from $\mathbb{P}^3$:

$$S = \left\{ \underbrace{4x^3 - 7x^2 - 5x + 6}_{q_1},\ \underbrace{2x^3 - 3x^2 - 7x + 3}_{q_2},\ \underbrace{10x^3 - 19x^2 + x + 15}_{q_3} \right\}$$

Let $p(x) = 2x^3 - 6x^2 + 20x + 3$.

Decide whether or not $p(x)$ is a member of $Span(S)$. Yes!

$$p(x) = c_1 q_1(x) + c_2 q_2(x) + c_3 q_3(x)$$

$$2x^3 - 6x^2 + 20x + 3$$

$$= c_1(4x^3 - 7x^2 - 5x + 6) = [4c_1 + 2c_2 + 10c_3]x^3$$

$$+ c_2(2x^3 - 3x^2 - 7x + 3) \quad + [\quad]x^2$$

$$+ c_3(10x^3 - 19x^2 + x + 15) \quad + [\quad]x$$

$$+ [\quad]\cdot 1$$

SOEs:

$$\left\{ \begin{array}{l} 2 = 4c_1 + 2c_2 + 10c_3 \\ \_\_\_ \\ \_\_\_ \\ \_\_\_ \end{array} \right.$$

$$\left[\begin{array}{ccc|c} 4 & 2 & 10 & 2 \\ -7 & -3 & -19 & -6 \\ -5 & -7 & 1 & 20 \\ 6 & 3 & 15 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 4 & 3 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Entire row of 0s : infinitely many solutions!
So yes!

Linear Independence of a Finite Set of Vectors

Definition: Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a set of vectors from a vector space $(V, \oplus, \odot)$. We say that S is *linearly independent* if and only if the only solution to the equation:

$$\boxed{(c_1 \odot \vec{v}_1) \oplus (c_2 \odot \vec{v}_2) \oplus \cdots \oplus (c_n \odot \vec{v}_n) = \vec{0}_V} \quad \text{DTE}$$

is the trivial solution $c_1 = 0$, $c_2 = 0$, $\ldots$, $c_n = 0$. As before, we will refer to this equation as a ***dependence test equation*** and sometimes just say "independent" to mean linearly independent. The opposite of being linearly independent is being linearly dependent, which means there is a non-trivial solution to the dependence test equation, that is, where at least one c_i is non-zero.

As before: a set of vectors $S \subseteq V$ is

- linearly dependent iff it's not linearly independent.
- linearly independent iff it's not linearly dependent.

Theorem: Let $(V, \oplus, \odot)$ be a vector space, and $\vec{v} \in V$. Then $S = \{\vec{v}\}$ is linearly independent ***if and only if*** $\vec{v} \neq \vec{0}_V$.

($\Rightarrow$) Contrapositive: Assume $\vec{v} = \vec{0}_V$. NTS: $S = \{\vec{v}\}$ is LD.

DTE: $c \odot \vec{v} = \vec{0}_V$. So since $\vec{v} = \vec{0}_V$, $c \odot \vec{v} = c \odot \vec{0}_V = \vec{0}_V$.

So $\exists$ ∞ly many sol to DTE. Any $c \in \mathbb{R}$ works! $\square$

($\Leftarrow$) exercise.

Theorem: Let $(V, \oplus, \odot)$ be a vector space, and $\vec{v}_1, \vec{v}_2 \in V$. Then $S = \{\vec{v}_1, \vec{v}_2\}$ is linearly independent ***if and only if*** $\vec{v}_1$ is not parallel to $\vec{v}_2$.

Pf exercise. $\square$

Theorem: Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a set of vectors from a vector space $(V, \oplus, \odot)$. Then: S is linearly ***dependent*** ***if and only if*** at least one vector (which, without loss of generality, we can set to be $\vec{v}_1$) is a linear combination of $\vec{v}_2, \vec{v}_3, \ldots, \vec{v}_n$, that is:

$$\vec{v}_1 = (r_2 \odot \vec{v}_2) \oplus (r_3 \odot \vec{v}_3) \oplus \cdots \oplus (r_n \odot \vec{v}_n),$$

for some scalars $r_2, r_3, \ldots, r_n \in \mathbb{R}$.

Pf exercise. $\square$ (Good test Q!)

A Sufficient Test for Independence of Sets of Polynomials

Theorem: Suppose $S = \{p_1(x), p_2(x), \ldots, p_k(x)\}$ is a set of polynomials from $\mathbb{P}^n$ with ***distinct*** degrees. Then S is linearly independent. In particular, the set $\{1, x, x^2, \ldots, x^n\}$ is linearly independent.

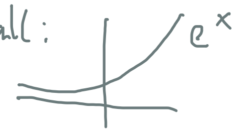
Pf. Next time.

Example: Consider the set $S = \{e^{-2x}, e^{x}, e^{5x}\}$.

LI or LD? DTE: $c_1 e^{-2x} + c_2 e^{x} + c_3 e^{5x} = z(x)$

Observation: $e^{ax} > 0$. Divide by e^{-2x}:

recall:



$$c_1 + c_2 e^{3x} + c_3 e^{7x} = z(x) \qquad (*)$$

$$\lim_{x \to -\infty} \left(c_1 + c_2 e^{3x} + c_3 e^{7x} \right) = \lim_{x \to -\infty} \left(z(x) \right)$$

Generalization:

$$c_1 + 0 + 0 = z(x) \longrightarrow \boxed{c_1 = 0}$$

$$c_2 e^{3x} + c_3 e^{7x} = z(x)$$

$$\lim_{x \to -\infty} \left(c_2 + c_3 e^{4x} \right) = \lim_{x \to -\infty} \left(z(x) \right)$$

$$c_2 + 0 = z(x) \longrightarrow \boxed{c_2 = 0}$$

$$c_3 e^{7x} = z(x)$$

$$\boxed{c_3 = 0}$$

Theorem: Suppose $S = \{e^{k_1 x}, e^{k_2 x}, \ldots, e^{k_n x}\}$, where $k_1 < k_2 < \cdots < k_n$ are n ***distinct*** real numbers. Then: S is linearly independent.

Pf Basically the same as in above example. $\square$