3.3 Linearity Properties for Infinite sets of Vectors

Goal Study $\boxed{\text{LCs}}$, $\boxed{\text{LI}}$ & LD, basis for infinite sets

$$S = \{\vec{v}_1, \vec{v}_2, \ldots\} \subseteq (V, \oplus, \odot)$$

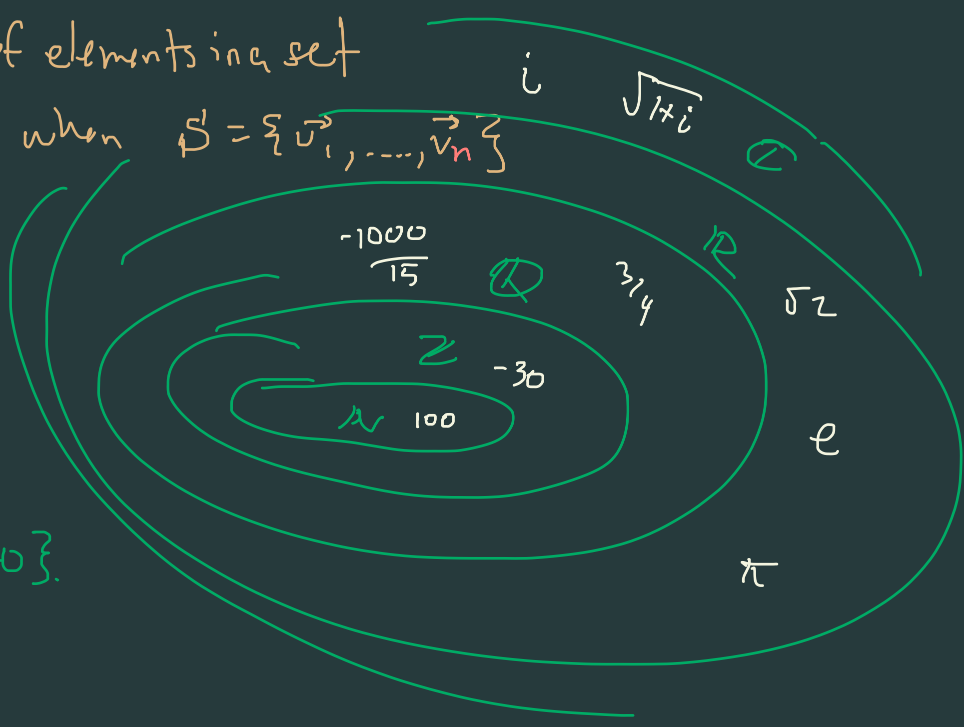
which are potentially infinite!

before need to study cardinality.

First, we need to do a deep study of infinite sets. Recall we previously introduced cardinality, but for finite sets. We now develop this concept rigorously & prove many surprising results.

Previously: Cardinality = # of elements in a set

ex: $\text{Card}(S) = n$ when $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$



Ex of Infinity sets

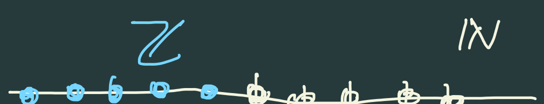
$\mathbb{N} = \{1, 2, 3, \ldots\}$

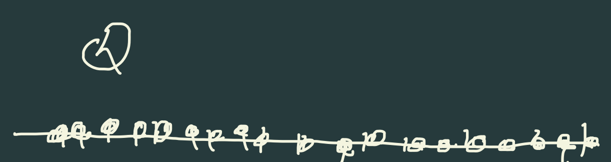
$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$

$\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$

Card( ) = ?

↳ precisely?





$\mathbb{Q}$ & $\mathbb{R}$ look same?

Def Cardinality. "intuitively: size of a set"

Let X & Y be two non-empty sets. We say X & Y have same cardinality if there's a function $\boxed{f: X \to Y}$ that's both 1-1 (injective) & onto (surjective).

In this case: write $\text{Card}(X) = \text{Card}(Y)$.
(or $|X| = |Y|$).

Remarks ① $\text{Card}(X)$ by itself is meaningless! Need two sets & comparing.

② Special cases (exemptions):

X is finite, just write $\text{Card}(X) = n$

to mean $\text{Card}(X) = \text{Card}(\{1, 2, \dots, n\})$.

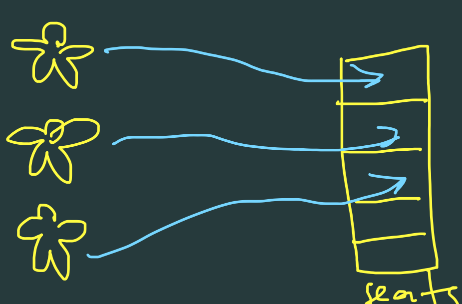
Ex $\text{Card}(X) = 3 = \text{Card}(\{a, b, c\})$.

③ Technically, this an "equivalence relations".

History Cantor: inventor of set theory.
↳ introduced cardinality
↳ extremely clever idea.
↳ "How to count without #s"
↳ Example: People & Movie Theatre (seats)

Cantor's Idea: have people sit & if there's no seats left & people standing then there's more people, etc....

called one-to-one correspondence



Card() < Card(seats)

Ex Paradoxical Examples w/ Infinite Sets:

$\mathbb{N}$ & $\mathbb{E} = \{\text{even \#s}\}$.

Show: $\text{Card}(\mathbb{N}) = \text{Card}(\mathbb{E})$.

$f: \mathbb{N} \to \mathbb{E}$, $f(n) = 2n$. $f(x) = 2x$

Is 1-1 & onto.

Notation & Terminology

- Countable sets: X is countable if $\text{Card}(X) = \text{Card}(\mathbb{N})$
- Uncountable sets: X is uncountable if it is not countable.
- Aleph Naught: $\aleph_0$ is by def $\boxed{\aleph_0 = \text{Card}(\mathbb{N})}$

Remark if a set X is countable this means there exists a function $f: \mathbb{N} \to X$ that's 1-1 & onto.

Theorem The following sets are all countable:

a) $\mathbb{N}$
b) $\mathbb{E}$ = even natural #s
c) $\mathbb{O}$ = odd natural numbers
d) $\mathbb{Z}$
e) $\mathbb{Q}$ (surprising!)

Pf (b) ✓ (c) exercise. (d) $f: \mathbb{N} \to \mathbb{Z}$

$$f\left(\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 0 & 1 & -1 & 2 & -2 & 3 & -3 & \cdots \end{array}\right)$$

1-1? yes, b/c each # in second list is only listed once!

Onto? yes, b/c eventually every $z \in \mathbb{Z}$ will be listed....

Pf cont (e) NTS $\text{card}(\mathbb{Q}) = \text{Card}(\mathbb{N}) = \aleph_0$

NTS: $f : \mathbb{N} \to \mathbb{Q}$ "snake argument"

$f : \mathbb{N} \to X$ where X contains $\mathbb{Q}$ (ie $\mathbb{Q} \subseteq X$)

[informally: $\text{card}(\mathbb{Q}) \leq \text{Card}(X)$ & $\text{card}(X) = \aleph_0$

& also $\mathbb{N} \subset \mathbb{Q}$ so $\aleph_0 \leq \text{Card}(\mathbb{Q}) \leq \aleph_0$]

X: $0, 1, -1, 2, -2, 3, -3, \ldots$ $(\mathbb{Z})$

$\frac{1}{2}, -\frac{1}{2}, \frac{2}{2}, -\frac{2}{2}, \frac{3}{2}, -\frac{3}{2}, \ldots,$

"snake argument"

$\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}, \frac{3}{3}, -\frac{3}{3}, \ldots$

$\square$

Bonus Theorem Let X, Y both countable sets.

Then so are:

a) $X \cup Y$

b) $X \times Y$

c) $X_1 \cup X_2 \cup \cdots \cup X_n$ (each X_i countable)

d) $X_1 \times X_2 \times \cdots \times X_n$ (")

e) $\bigcup_{\substack{i=1 \\ i \in \mathbb{N}}}^{\infty} X_i$ still countable!

Theorem (Cantor) "Cantor's Diagonalization Argument"

$\mathbb{Q}$ and $\mathbb{R}$ don't have the same cardinality!

Pf By contradiction. Assume that $\mathbb{Q}$ and $\mathbb{R}$ have same cardinality. So, by previous theorem $\mathbb{N}$ and $\mathbb{R}$ have same cardinality. By def of cardinality, there's a function $f : \mathbb{N} \to \mathbb{R}$.

Thm

$\mathbb{N}$ n	$\mathbb{R}$ x
1	$\boxed{3}.4156\ldots$
2	$0.\boxed{7}89\ldots$
3	$-11.1\boxed{1}1\ldots$
4	$-2.50\boxed{0}0\ldots$
5	$0.898\boxed{9}1131113\ldots$
6	$\vdots$
$\vdots$	$\vdots$

- Clever Idea #1 Use decimal expansions!

$$3.415\ldots = n + \frac{x_1}{10} + \frac{x_2}{100} + \frac{x_3}{1000} + \cdots$$

- Clever Idea #2 build a real # not on this list

r = it differs from every # on our list by a digit!

ex whole # part = take 2
tenths part = take 6
hundreths part = take 1
thousandths part = take 8 .....

so by "construction"
$r \neq x$ for any x in list
$(x = f(n) \ \forall n \in \mathbb{N})$

This is a contradiction so our assumption that $\mathbb{R}$ is countable is wrong!
so $\mathbb{R}$ is uncountable. $\square$

Terminology $\aleph_1 = \text{Card}(\mathbb{R})$ $\aleph_1$ = "first uncountable infinity"

Bonus Theorem def $\mathcal{P}(X)$ = power set of X
= set of all subsets of X.

$$\text{Card}(\mathcal{P}(\mathbb{N})) = \text{Card}(\mathbb{R}).$$

Rmk If $\text{Card}(X) = \aleph_1$, X is called uncountable.

def
- $\text{Card}(X) < \text{Card}(Y)$ means $\exists f: X \to Y$ 1-1 but no such map can also be onto.
- $\text{Card}(X) \leq \text{Card}(Y)$ means $\exists f: X \to Y$ that's 1-1 (can be onto or not)
- $\text{Card}(Y) > \text{Card}(X)$
- $\text{Card}(Y) \geq \text{Card}(X)$.

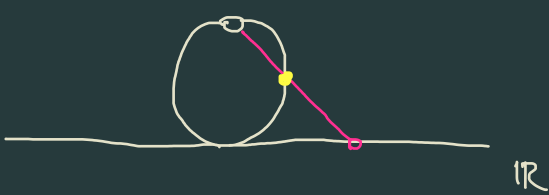
Theorem $\aleph_1 = \text{Card}(\mathbb{R}) = \text{Card}([0,1]) = \text{Card}([a,b])$ $\quad (a < b)$

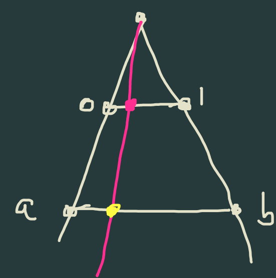
$= \text{Card}((0,1)) = \text{Card}((a,b))$

$= \text{Card}([0,1)) = \text{Card}([a,b))$

$= \text{Card}((0,\infty)) = \text{Card}([B,\infty))$.

Pf by Picture





$f: \mathbb{R} \to (a,b)$

$\tan: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$. $\square$

back to LA

Indexing Sets

past: $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ finit list, $\vec{v}_i$ i is an index.

now: $S = \{\vec{v}_i \mid i \text{ index from a set } I\}$ I for index / not interval

Assume $I \subseteq \mathbb{R}$, I not empty $(I \neq \emptyset)$

$I = \{1, 2, \dots, n\}$ same as before.

$I = \mathbb{N}$, or $I = \mathbb{R}$

Examples $\mathbb{P}^n = \{\text{polynomials degree} \leq n\} \subseteq \mathcal{F}(\mathbb{R}, \mathbb{R})$

$S = \{1, x, x^2, \dots, x^n\}. = \{x^i \mid i \in \{1, 2, \dots, n\}\}$

Ex $\mathbb{P} = \{\text{all polynomials of all degrees}\}$.

$S = \{1, x, x^2, \dots, x^n, x^{n+1}, \dots\} = \{x^i \mid i \in \mathbb{N}\} = \{x^n \mid n \in \mathbb{N}\}$

sometime we'll use $\mathbb{N}$ that includes 0.

$E = \{1, x^2, x^4, \dots, x^{100}, \dots\} = \{x^{2n} \mid n \in \mathbb{N}\}$.

Say: S, E are countably indexed (in particular, countable).

Ex $S_1 = \{e^{kx} \mid k \in \mathbb{Z}\}$ (previously $\{e^{-2x}, e^x, e^{3x}\}$ (LI set))

↳ countably indexed.

$S_2 = \{e^{kx} \mid k \in \mathbb{Q}\}$ $\quad e^{-x/2} \in S_2$ but not S_1.

$S_3 = \{e^{kx} \mid k \in \mathbb{R}\}$

↳ uncountably indexed.

$S_1 \subsetneq S_2 \subsetneq S_3$

Linear Combinations

(previously: $S = \{\vec{v}_1, \dots, \vec{v}_n\}$: LC: $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$
where $c_i \in \mathbb{R}$, $\vec{v}_i \in \mathbb{R}^k$.)

Mimic this but S can be infinite now (countably or uncountably infinite).

Let $\quad I$. When I is finite, $\operatorname{Card}(I) = n$, say $n \in \mathbb{N}$.

Then $S = \{\vec{v}_i \mid i \in I\}$ we can list as before $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$.

- Now, I finite or infinite: list a finite subset of I as follows:

$i_1, i_2, \dots, i_n$ WLOG with out loss of generality

$\rightarrow i_1 < i_2 < \dots < i_n$

(reminder $I \subseteq \mathbb{R}$ so these are all real #s).

Write $\{\vec{v}_{i_1}, \vec{v}_{i_2}, \dots, \vec{v}_{i_n}\}$ finite subset of $(V, \oplus, \odot)$

Def LINEAR COMBINATIONS.

Let $(V, \oplus, \odot)$ be a vector space.

Let $S = \{\vec{v}_i \mid i \in I\}$ some $I \subseteq \mathbb{R}$, $I \neq \emptyset$,

The a linear combination of vectors of S is constructed as follows:

a) choose a finite set of vectors in S:

$$\vec{v}_{i_1}, \vec{v}_{i_2}, \dots, \vec{v}_{i_n} \in S, \quad w/ \quad i_1 < i_2 < \dots < i_n.$$

b) choose a finite set of coefficients:

$$c_1, c_2, \dots, c_n \in \mathbb{R}.$$

c) a linear combination is:

$$\boxed{c_1 \odot \vec{v}_{i_1} \oplus c_2 \odot \vec{v}_{i_2} \oplus \dots \oplus c_n \odot \vec{v}_{i_n}} \quad LC$$

def $\boxed{\text{Span}(S)} = \{$ all LCs of all possible finite subsets of $S\}$

Remarks ① we don't define "infinite" LCs : in our class we always form finite LC of vectors.

② if $I = \mathbb{N}$ or even $I = \mathbb{R}$

Span(S) unfathomably complex!

Thm *Usefull!*

Special Case I is countable, S is countably indexed:

- Write $S = \{\vec{v}_1, \vec{v}_2, \ldots\}$. Then LCs are of the form:

$$c_1 \odot \vec{v}_1 \oplus c_2 \odot \vec{v}_2 \oplus \cdots \oplus c_k \odot \vec{v}_k$$

- $$\mathrm{Span}(S) = \left\{ c_1 \odot \vec{v}_1 \oplus \cdots \oplus c_k \odot \vec{v}_k \;\middle|\; \begin{array}{l} \text{some } k \in \mathbb{N}. \\ c_1, \ldots, c_k \in \mathbb{R},\ \vec{v}_i \in S \end{array} \right\}$$

Pf Next time.

Ex $\boxed{S = \{x^n \mid n \in \mathbb{N}_0\}} = \{1, x, x^2, \ldots\}$ "monomials"

$\to \mathbb{N} = \{0, 1, 2, \ldots\}$

$\to \mathbb{N} = \{1, 2, 3, \ldots\}$

$\mathbb{N}_0 = \{0, 1, 2, \ldots\}$

- $S \not\subseteq \mathbb{P}^n$

- LCs By theorem of countably indexed sets

Any $p(x) \in \mathrm{Span}(S)$: for some $k \in \mathbb{N}$:

$$p(x) = c_0 \cdot 1 + c_1 \cdot x + c_2 \cdot x^2 + \cdots + c_k \cdot x^k$$

So Span(S) contains $\mathbb{P}^k$, true for all $k \in \mathbb{N}$

$$\mathbb{P}^0 \subseteq \mathbb{P}^1 \subseteq \mathbb{P}^2 \subseteq \cdots \subseteq \mathbb{P}^k \subseteq \cdots \subseteq \mathrm{Span}(S).$$

(bonus notation: $\mathbb{P} = \mathbb{P}^0 \cup \mathbb{P}^1 \cup \mathbb{P}^2 \cup \ldots \cup \mathbb{P}^n \cup \ldots$

$$= \bigcup_{k \in \mathbb{N}} \mathbb{P}^k \;)$$

<u>Pf of Thm</u> (I is countable)

A finite set of vectors from S has the form:

$$\{ \vec{v}_{i_1}, \vec{v}_{i_2}, \ldots, \vec{v}_{i_n} \} \quad \text{where } i_1 < i_2 < \ldots < i_n \quad (\text{WLOG})$$

so a LC:

$$r_1 \odot \vec{v}_{i_1} \oplus \ldots \oplus r_n \odot \vec{v}_{i_n} \quad (\text{scalars } r_i \in \mathbb{R}) \qquad (*)$$

(Idea: make a larger LC w/ $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ & new scalars $c_j = 0$ when $\vec{v}_j$ doesn't show up in $(*)$ but is r_j when $\vec{v}_{i_j}$ does)

Let

$$c_j = \begin{cases} r_j, & \text{if } j = i_1, i_2, \ldots, i_n \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$r_1 \odot \vec{v}_{i_1} \oplus \ldots \oplus r_n \odot \vec{v}_{i_n} = c_1 \odot \vec{v}_1 \oplus c_2 \odot \vec{v}_2 \oplus \ldots \oplus c_k \odot \vec{v}_k .$$

Result follows. $\square$

<u>EX</u> $\{ \vec{v}_2, \vec{v}_5, \vec{v}_7 \}$ w/ $i_1 = 2,\ i_2 = 5,\ i_3 = 7$ $\quad \{ \vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_7 \}$

Then

$$c_j = \begin{cases} r_j, & \text{if } j = 2, 5, 7 \\ 0, & \text{otherwise.} \end{cases}$$

So:

$$r_1 \vec{v}_2 + r_2 \vec{v}_5 + r_3 \vec{v}_7 = 0 \vec{v}_1 + c_2 \vec{v}_2 + 0 \vec{v}_3 + 0 \vec{v}_4 + c_5 \vec{v}_5 + 0 \vec{v}_6 + c_7 \vec{v}_7$$

Ex $S = \{x^n \mid n \in \mathbb{N}\}$, ie $p(x) = x^n$ polynomial/monomial

$S \notin \mathbb{P}^n$

S countably indexed. By our theorem: a $\underline{LC}$:

$$c_0 \cdot 1 + c_1 \cdot x + c_2 \cdot x^2 + \cdots + c_k \cdot x^k \quad (k \in \mathbb{N})$$

a poly of degree n!

So: $\text{Span}(S) =$ contains $\mathbb{P}$.

$$= \bigcup_{n \in \mathbb{N}} \mathbb{P}^n$$

!

- $\mathbb{P}$ is a $\underline{VS}$: why? Consider $f, g \in \mathbb{P}$.

Then $\deg(f) \in \mathbb{N}$, $\deg(g) \in \mathbb{N}$ so $\exists\, n, m \in \mathbb{N}$, where $n = \deg(f)$, $m = \deg(g)$.

$$f(x) = c_0 \cdot 1 + c_1 \cdot x + \cdots + c_n x^n$$

$$g(x) = d_0 \cdot 1 + d_1 \cdot x + \cdots + d_m \cdot x^m$$

Let $k = \max(n, m) = \begin{cases} n & \text{if } n \geq m \\ m & \text{if } n < m \end{cases}$

$(f+g)(x) =$ a poly of degree k so $f+g \in \mathbb{P}^k \subseteq \mathbb{P}$.

Ex $S_3' = \{e^{kx} \mid k \in \mathbb{R}\} \subseteq \mathcal{F}(\mathbb{R}, \mathbb{R})$ "uncountably indexed"

a LC: pick $\underline{\text{finite}}$ set of real #s: $k_1 < k_2 < \cdots < k_n \in \mathbb{R}$.

(LC) $c_1 e^{k_1 x} + c_2 e^{k_2 x} + \cdots + c_n e^{k_n x}$ Ex: $c_1 e^{\sqrt{2}x} + c_2 e^{3x} + c_3 e^{\pi x}$. $(c_i \in \mathbb{R})$

Linearly Independent Sets

- Suppose $S = \{ \vec{v}_i \mid i \in I \} \subseteq (V, \oplus, \odot)$, $I \neq \emptyset$.

def say S is **linearly independent** if every finite subset of I gives LI vectors $\{ \vec{v}_{i_1}, \vec{v}_{i_2}, \ldots, \vec{v}_{i_n} \}$:

$\boxed{\text{DTE}}$ $$\boxed{c_1 \odot \vec{v}_{i_1} \oplus c_2 \odot \vec{v}_{i_2} \oplus \cdots \oplus c_n \odot \vec{v}_{i_n} = \vec{0}_V}$$

has the trivial solution $\vec{c} = \langle c_1, c_2, \ldots, c_n \rangle = \langle 0, 0, \ldots, 0 \rangle$ as the only solution.

$\boxed{\text{Thm}}$ When S is countable, $S = \{ \vec{v}_i \mid i \in I \} = \{ \vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots \}$

S is $\boxed{\text{LI}}$ $\boxed{\text{iff}}$ $\forall k \in \mathbb{N}$:

$$\left[c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_k \vec{v}_k = \vec{0}_V \right] \Longrightarrow \left[c_1 = \cdots = c_k = 0 \right]$$

Ex $S = \{ x^n \mid n \in \mathbb{N} \}$ ∞

$S_n = \{ 1, x, \ldots, x^n \}$ (finite)

is LI: $\forall n$:

each poly has a distinct degree. (§3.2).

So S is a basis for

$\text{Span}(S) = \mathbb{P}$

Ex $S_3 = \{ e^{kx} \mid k \in \mathbb{R} \}$

- is S_3 LI?

Choose finite indices: $k_1 < k_2 < \cdots < k_n$

form:

$$S_n = \{ e^{k_1 x}, e^{k_2 x}, \ldots, e^{k_n x} \}.$$

Is S_n LI? Yes! Did this in §3.2.

S_3 is a basis for

$\text{Span}(S_3)$.