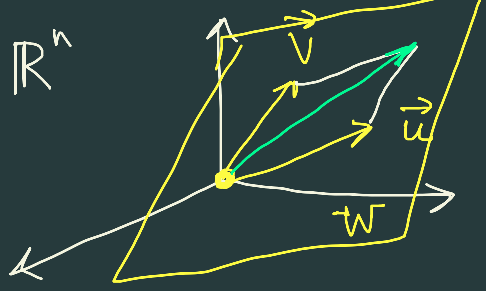
Section 3.4: Subspaces, Basis, Dimension

In $\mathbb{R}^n$ subspace: $W \trianglelefteq \mathbb{R}^n$: ① $W \neq \emptyset$
② CA: closed under addition
③ CSM: closed under scaler mult.



Thms of note

1) $\text{Span}(\{\vec{v}_1, \dots, \vec{v}_K\})$ is a subspace
2) $\vec{0}_V \in W$ if $W \trianglelefteq \mathbb{R}^n$
3) $\dim(W) \leq n$ when $W \trianglelefteq \mathbb{R}^n$

In $(V, \oplus, \odot)$:

def A subset $W \subseteq (V, \oplus, \odot)$ satisfies:

① W is non-empty.

② Closed under Vector Addition (CVA) : $\forall \vec{w}_1, \vec{w}_2 \in W \Rightarrow \vec{w}_1 \oplus \vec{w}_2 \in W$

③ Closed under Scalar Mult. (CSM) : $\forall r \in \mathbb{R}, \forall \vec{w} \in W \Rightarrow r \odot \vec{w} \in W$

As before call V the ambient space.

$\boxed{\text{Thm}}$ W is a subspace of $(V, \oplus, \odot)$ $\boxed{\text{iff}}$ $(W, \oplus, \odot)$ is a Vector Space.

Say $\oplus, \odot$ is "inherited from V".

Pf ($\Leftarrow$) ✓

($\Rightarrow$) • VSA: 3, 4, 7, 8, 9, 10 : automatically true b/c $W \subseteq V$.

↪ again : "inheritance"

• VSA1 ✓ same as CVA
• VSA2 ✓ same as CSM

• $\vec{0}_W$? VSA5

• $\forall \vec{w} \in W$, $-\vec{w} \in W$? VSA6

- $\vec{0}_W = \vec{0}_V$:

Since W is non-empty, $\exists\ \vec{\omega} \in W$.

By CSM, since $0 \in \mathbb{R}$: $0 \odot \vec{\omega} \in W$.

Since: $0 \odot \vec{\omega} = \vec{0}_V$ (zero Properties)

so $\vec{0}_V \in W$. By Uniqueness of zero: $\vec{0}_W = \vec{0}_V$.

- Add Inverses:

Since $W \subseteq V$, $\forall \vec{\omega} \in W$, $\vec{\omega} \in V$, so VSA6 on V:

$\exists\ -\vec{\omega} \in V$ so $\vec{\omega} \oplus (-\vec{\omega}) = \vec{0}_V = (-\vec{\omega}) \oplus \vec{\omega}$.

By CSM: $-\vec{\omega} = (-1) \odot \vec{\omega} \in W$. So $-\vec{\omega} \in W$.

prop. proved previously,

$\square$

☆ key in LA ☆

Thm "Span is a subspace"

- Assume $S = \{\vec{v}_i \mid i \in I\} \subseteq (V, \oplus, \odot)$ where $I \subseteq \mathbb{R}$ (non-empty)
- Let $\boxed{W = \text{Span}(S)}$. **Thm** $W \trianglelefteq V$, ie is a subspace.

Pf Since I is non-empty, S is non-empty, & $S \subseteq \text{Span}(S)$.

so $\text{Span}(S)$ is non-empty. so ① is true in def of subspace.

CVA Let $\vec{\omega}_1, \vec{\omega}_2 \in W$.

Since $\vec{\omega}_1, \vec{\omega}_2$ are LCs of S:

$\exists$ finite indexing sets: $i_1 < i_2 < \cdots < i_n$: so that.

$i_1' < i_2' < \cdots < i_m'$:

so that:

$$\vec{w}_1 = r_1 \odot \vec{v}_{i_1} \oplus \cdots \oplus r_n \odot \vec{v}_{i_n} \qquad (r_1, r_2, \ldots, r_n \in \mathbb{R})$$

$$\vec{w}_2 = s_1 \odot \vec{v}_{i_1'} \oplus \cdots \oplus s_m \odot \vec{v}_{i_m'} \qquad (s_1, \ldots, s_m \in \mathbb{R})$$

Let $S_1 = \{i_1, \ldots, i_n\}$, $S_2 = \{i_1', \ldots, i_m'\}$, let

$$S_3 = S_1' \cup S_2$$
$$= \{l_1, l_2, \ldots, l_k\} \text{ where } k \leq n+m$$

(note: k could be $< n+m$ b/c S_1' & S_2 could share indices).

Define $c_j, d_j \in \mathbb{R}$:

$$c_j = \begin{cases} r_j & \text{if } j \in S_1 \\ 0 & \text{otherwise} \end{cases} \qquad d_j = \begin{cases} s_j & \text{if } j \in S_2 \\ 0 & \text{otherwise.} \end{cases}$$

$$\vec{w}_1 = r_1 \vec{v}_{i_1} + \cdots + r_n \vec{v}_{i_n} = c_1 \vec{v}_{l_1} + c_2 \vec{v}_{l_2} + \cdots + c_k \vec{v}_{l_k}$$

$$\vec{w}_2 = s_1 \vec{v}_{i_1'} + \cdots + s_m \vec{v}_{i_m'} = d_1 \vec{v}_{l_1} + d_2 \vec{v}_{l_2} + \cdots + d_k \vec{v}_{l_k}$$

Then

$$\vec{w}_1 \oplus \vec{w}_2 = \underbrace{(c_1+d_1) \odot \vec{v}_{l_1} \oplus (c_2+d_2) \odot \vec{v}_{l_2} \oplus \cdots \oplus (c_k+d_k) \odot \vec{v}_{l_k}}$$

Is this a LC of vectors from S'?

yes! a finite LC of vectors from S'!

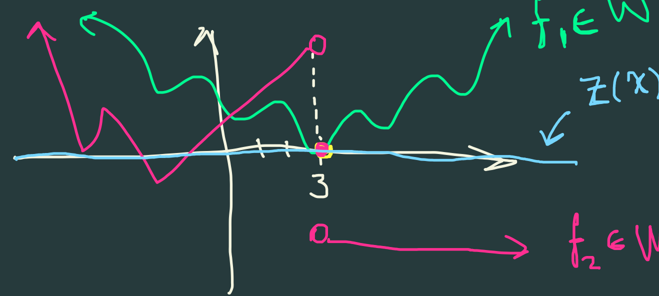
$\in \mathrm{Span}(S') = W$.

CSM $\forall r \in \mathbb{R}, \vec{w}_1 \in W$: $r \odot \vec{w}_1 = r \odot (r_1 \odot \vec{v}_{i_1} \oplus \cdots \oplus r_n \odot \vec{v}_{i_n}) \overset{\text{VSA 8,9}}{=} (rr_1) \odot \vec{v}_{i_1} \oplus \cdots \oplus (rr_n) \odot \vec{v}_{i_n}$

as before. $\square$

Ex $\mathcal{F}(\mathbb{R},\mathbb{R}) = \{ f: \mathbb{R} \to \mathbb{R} \mid \text{function} \}. = V$

$W = \{ f \in V \mid f(3) = 0 \}.$



show W is a subspace:

1) W is non-empty:

note: $Z(x) = 0 \ \forall x \in \mathbb{R}$ satisfies $Z(3) = 0$. So $Z(x) \in W$.

2) CVA: let $f, g \in W$. Then $f(3) = 0$, $g(3) = 0$.

So $\underbrace{(f+g)(x)}_{f \oplus g} = f(x) + g(x) \ \forall x \in \mathbb{R}$ (pointwise-principle)

$(f+g)(3) = f(3) + g(3) = 0 + 0 = 0.$ So $f + g \in W$.

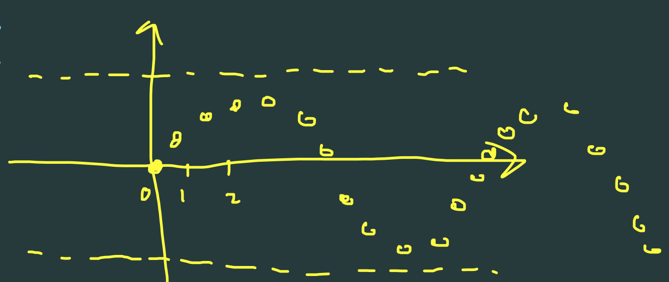
3) CSM: $kf \in W$ since $(kf)(3) = k \cdot f(3) = k \cdot 0 = 0.$

Ex $W_1 = \{ f \in \mathcal{F}(\mathbb{R},\mathbb{R}) \mid f(3) = 3 \}$ is W_1 a subspace?

Ex $\ell_\infty = \{$ all bounded sequences $\}$

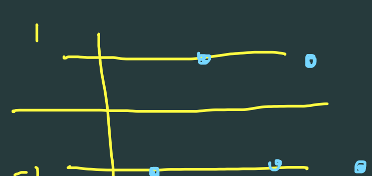
$= \{ \langle x_n \rangle_{n=1}^{\infty} \mid x_n \in \mathbb{R}, \text{ bounded}: \exists M \in \mathbb{R} \ (M < \infty) : |x_n| \le M \ \forall n \in \mathbb{N} \}$

Ex



Ex $\langle \frac{1}{n} \rangle_{n=1}^{\infty} = \langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \rangle \in \ell_\infty$

Ex $\langle (-1)^n \rangle_{n=1}^{\infty}$



$W = \{$ sequences w/ only a finite number non-zero #s $\}$.

$c_0 = \{$ sequences "eventually" zero $\}$.

$c_0 = W = \{\text{sequence eventually zero}\}$. NTS $c_0 \subseteq \ell_\infty$ & $c_0 \trianglelefteq \ell_\infty$.

- $\langle x_n \rangle_{n=1}^{\infty} \in c_0 \Rightarrow \langle x_n \rangle_{n=1}^{\infty} \in \ell_\infty$:

  $\hookrightarrow$ $\langle x_1, x_2, x_3, \ldots\ldots, x_n, 0, 0, 0, \ldots\ldots \rangle$ is it bounded?

  yes! $M = \max\{x_1, \ldots, x_n\}$. Then $|x_n| \leq M \ \forall n \in \mathbb{N}$.

- CVA: $\langle x_n \rangle, \langle y_n \rangle \in c_0$:

  $\hookrightarrow$ $\langle x_1, x_2, \ldots\ldots, x_n, 0, 0, 0, \ldots\ldots \rangle$

  $\langle y_1, y_2, \ldots, y_m, 0, 0, 0, \ldots\ldots \rangle$

Then let $k = \max(n, m)$.

$\langle x_n \rangle + \langle y_n \rangle = \langle x_1 + y_1, x_2 + y_2, \ldots\ldots, x_k + y_k, 0, 0, 0, \ldots\ldots \rangle \in c_0$.

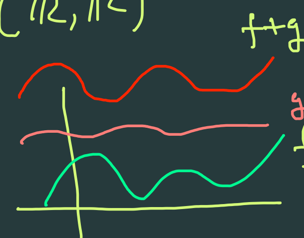
so when $n \geq k$, it is zero!

Big Ex on $\mathcal{F}(\mathbb{R}, \mathbb{R})$

- $\mathcal{C}(\mathbb{R}, \mathbb{R}) = \{ f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) \mid f \text{ is continuous} \} \trianglelefteq \mathcal{F}(\mathbb{R}, \mathbb{R})$

  ex: CVA: f, g are continuous:

  $f+g$ is continuous (Calc 1)



- $\mathcal{D}(\mathbb{R}, \mathbb{R}) = \{ f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) \mid f \text{ is differentiable} \}$

  ex: CVA: f, g are differentiable: $f+g$ is diff (Calc 1)

  Moreover: $\underline{(f+g)' = f' + g'}$ (Calc 1)

- $\mathcal{C}^1(\mathbb{R}, \mathbb{R}) = \{ f \text{ has continuous derivative, i.e. } f' \text{ is a continuous function} \}$.
- $\mathcal{C}^n(\mathbb{R}, \mathbb{R}) = \{ f, f', f'', \ldots, f^{(n)} \text{ \& } f^{(n)} \text{ is continuous} \}$.

- $C^\infty(\mathbb{R},\mathbb{R}) = \{$ infinitely differentiable functions $\}$

e^x, $\sin(x)$, $\cos(x)$, $\frac{1}{1+x^2}$

★ BASIS ★ key topic in LA ★

def A set of vectors $B \subseteq (V, \oplus, \odot)$ is a basis for V if B is non-empty, B is LI, & $\mathrm{Span}(B) = V$.

Ex $S = \{x^n \mid n \in \mathbb{N}\} = \{1, x, x^2, \dots\}$ is a basis for $\mathbb{P}$.

$S_n = \{1, x, x^2, \dots, x^n\}$ is a basis for $\mathbb{P}^n$. **Note** $\mathbb{P}^n \trianglelefteq \mathbb{P}$.

★**Ex**★ • $M_{m\times n}(\mathbb{R}) = \{$all $m\times n$ matrices w/ entries in $\mathbb{R}\}$.

• lets build a basis for $M_{m\times n}$.

$E = \{E_{ij} \mid i \in \{1,2,\dots,m\},\ j \in \{1,2,\dots,n\}\}$. has $m\cdot n$ matrices.

$$E_{11} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ & & & 0 & & \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ & & & 0 & & \end{bmatrix}, \dots$$

$$E_{ij} = \begin{array}{c} \\ \\ i \\ \\ \end{array}\begin{array}{c} \qquad j \\ \begin{bmatrix} 0 & & 0 & & \\ & & \vdots & & \\ 0 & \cdots & 1 & \cdots & 0 \\ & & \vdots & & \\ & & 0 & & \end{bmatrix}\end{array}, \dots, \quad E_{mn} = \begin{array}{c} \\ \\ \\ m \end{array}\begin{array}{c} \qquad\qquad n \\ \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & 0 & & \vdots \\ & & & 0 \\ 0 & 0 & \cdots\ 0 & 1 \end{bmatrix}\end{array}$$

Is E LI?

$\boxed{\text{DTE}}$ $c_{11}E_{11} + c_{12}E_{12} + \cdots + c_{mn}E_{mn} = [0]_{m\times n}$

- look at 1,1 entry: LHS: $c_{11}+0+0+\cdots+0$. RHS 0

  $\hookrightarrow c_{11}=0.$

- look at i,j entry: LHL: $0+0+\cdots+c_{ij}+0+\cdots+0$, RHS 0

  $\hookrightarrow c_{ij}=0$

- all coefficients $c_{11}, c_{12}, \ldots, c_{ij}, \ldots, c_{mn}=0$. LI!

- Is E spanning?

Yes, tedius:

$$A=(a_{ij})_{m\times n}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ & & & a_{mn}\end{bmatrix}=a_{11}\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ & & & & \end{bmatrix}+a_{12}\begin{bmatrix} 0 & 1 & 0 & \cdots \\ & & & \end{bmatrix}+\cdots$$

$$=a_{11}E_{11}+a_{12}E_{12}+\cdots+a_{mn}E_{mn}$$

so A is a LC of matrices in E!

$\boxed{\text{Thm}}$ $S=\{\vec{v}_i \mid i\in I\}$, Index $\neq \phi$, $\subseteq (V,\oplus,\odot)$.

S is a basis $\boxed{\text{iff}}$ every vector $\vec{v}\in V$ can be uniquely represented as a LC of a finite subset of vectors

$\{\vec{v}_{i_1},\vec{v}_{i_2},\ldots,\vec{v}_{i_n}\}$ from S:

$$\boxed{\vec{v}=c_1\vec{v}_{i_1}+c_2\vec{v}_{i_2}+\cdots+c_n\vec{v}_{i_n}}$$

$c_1,c_2,\ldots,c_n$ are unique & $\vec{v}_{i_1},\ldots,\vec{v}_{i_n}$ are unique.

Pf ($\Rightarrow$) Assume that S is a basis for V.

So S is LI & span$(S)=V$.

$S_1 = \{i_1, \dots, i_n\}$.

So: since S' spans V $\therefore$ $\exists$ finite set indices $i_1 < i_2 < i_3 < \dots < i_n \subseteq I$.
& $\exists\, c_1, \dots, c_n \in \mathbb{R}$: Any $\vec{v} \in V = \operatorname{Span}(V)$

$$\vec{v} = c_1 \odot \vec{v}_{i_1} \oplus \dots \oplus c_n \odot \vec{v}_{i_n}.$$

NTS this is unique.

Argue by contradiction. Assume not. There's another set of indices $S_2' = \{j_1, \dots, j_m\} \subseteq I$ (increasing), $\exists\, d_1, \dots, d_m$

$$\vec{v} = d_1 \odot \vec{v}_{j_1} \oplus \dots \oplus d_m \odot \vec{v}_{j_m}.$$

☆ Let $S_3' = S_1 \cup S_2 = \{l_1, l_2, \dots, l_k\}$, $l_1 < l_2 < \dots < l_k$, $k \le n+m$

Let $r_j = \begin{cases} c_j & j = i_1, \dots, i_n \\ 0 & \text{otherwise} \end{cases}$ $\quad s_j = \begin{cases} d_j & \text{if } j = j_1, \dots, j_m \\ 0 & \text{otherwise.} \end{cases}$

$$\vec{v} = r_1 \vec{v}_{l_1} + r_2 \vec{v}_{l_2} + \dots + r_k \vec{v}_{l_k}$$
$$\vec{v} = s_1 \vec{v}_{l_1} + s_2 \vec{v}_{l_2} + \dots + s_k \vec{v}_{l_k}$$

so:

(☆) $$\vec{0}_V = \vec{v} - \vec{v} = (r_1 - s_1)\vec{v}_{l_1} + (r_2 - s_2)\vec{v}_{l_2} + \dots + (r_k - s_k)\vec{v}_{l_k}$$ (VSA 4, 7)

Since $S = \{\vec{v}_i \mid i \in I\}$ is Linearly Independent, (☆) implies

$$r_1 - s_1 = 0, \dots, r_k - s_k = 0 \iff \boxed{r_1 = s_1, \dots, r_k = s_k}.$$

So not only do the coefficients match so too do the vectors!

($\Longleftarrow$) Assume any $\vec{v}$ can be written uniquely.
NTS S is LI & $\operatorname{Span}(S) = V$.

Observe: (☆) eq in statement of thm is exactly $\operatorname{Span}(S) = V$

- Set-up DTE to check LI:

Let $S_1 = \{ i_1, \dots, i_n \} \subseteq I$ any finite set of indices in I.

Consider

$$(*) \qquad c_1 \vec{v}_{i_1} + c_2 \vec{v}_{i_2} + \dots + c_n \vec{v}_{i_n} = \vec{0}_V, \quad (c_1, \dots, c_n \in \mathbb{R})$$

- By Uniqueness Assumption: $\vec{0}_V \in V$: so

$$(**) \qquad \vec{0}_V = 0 \cdot \vec{v}_{i_1} + 0 \cdot \vec{v}_{i_2} + \dots + 0 \vec{v}_{i_n} ,$$

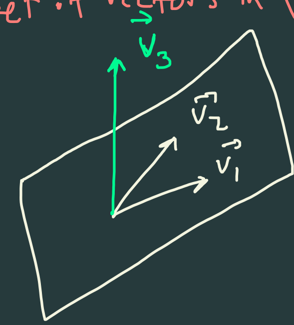
Uniquess property & $(*)$ & $(**)$: $c_1 = 0, \dots, c_n = 0$!

So S' is LI. $\square$

$\boxed{\text{Thm}}$ ☆ Extension Theorem ☆

$S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be a finite, linearly independent set of vectors in V.

Suppose $\vec{v}_{n+1} \notin \text{Span}(S)$. $\boxed{\text{Then}}$



$$S' = S \cup \{\vec{v}_{n+1}\} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n, \vec{v}_{n+1}\}$$

is still $\boxed{\text{LI}}$!

Sketch DTE: $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n + c_{n+1}\vec{v}_{n+1} = \vec{0}_V$

Case $c_{n+1} = 0$

Case $c_{n+1} \neq 0$

□

$\boxed{\text{Thm}}$ $\boxed{☆\text{DEEP}☆}$ Every vector space has a basis.

Pf way outside our scope. □

DIMENSION

$(V, \oplus, \odot)$, $W \trianglelefteq V$

def dimension of V is $\text{Card}(B)$, where B is a basis.

Recall when $\text{Card}(B)$ is finite, we call V a finite-dimensional vector space.

Otherwise, say V is infinite-dimensional.

Agree: $\{\vec{0}_V\}$ has no basis (or basis $= \emptyset$)

$\boxed{\text{Q}}$ Can a vector space have 2 different dimensions? NO!

If $\mathcal{B}$ & $\mathcal{B}'$ are two bases for $\mathcal{V}$ then $Card(\mathcal{B}) = Card(\mathcal{B}')$

↳ dimension is unique!

$\boxed{\text{Thm}}$ Dep/Ind sets from Spanning sets Thm (Gen $\mathcal{V}$ version)

Suppose $\mathcal{S} = \{\vec{w}_1, \dots, \vec{w}_n\}$, $Card(\mathcal{S}) = n$. Let $\mathcal{W} = Span(\mathcal{S})$.

Suppose we randomly choose m vectors $\vec{u}_i$'s from $\mathcal{W}$ & form:

$$L = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\} \subseteq \mathcal{W}$$

Then:

- if $\boxed{m > n}$ then L is LD.
- if L is LI then $\boxed{m \leq n}$

Pf Basically, same as before. □

$\boxed{\text{Thm}}$ Any two bases from a finite-dimensional vector space have exactly the same # of vectors.

↳ $\dim(\mathcal{V})$ is unique.

Pf
- $\mathcal{B}$ is a basis for $\mathcal{V}$ & so is $\mathcal{B}'$.
- $Span(\mathcal{B}) = \mathcal{V} = Span(\mathcal{B}')$
- Get: $\overset{LI}{\mathcal{B}} \subseteq Span(\mathcal{B}')$ & $\overset{LI}{\mathcal{B}'} \subseteq Span(\mathcal{B})$.
- Apply Dep/Ind Set from Spanning sets theorem:

$$Card(\mathcal{B}) \leq Card(\mathcal{B}') \quad \& \quad Card(\mathcal{B}') \leq Card(\mathcal{B})$$

so =

□

Rmk This theorem is true even for infinite dimensional V.S's.

Ex • $\mathcal{B}_n = \{1, x, x^2, \dots, x^n\}$ is a basis for $\mathbb{P}^n$

- $B = \{1, x, x^2, \ldots, x^n, \ldots\} = \{x^n \mid n \in \mathbb{N}\}$ is a basis for $\mathbb{P}$.

  $\dim(\mathbb{P})$ = infinite-dimensional $= \aleph_0$

- $S_3 = \{e^{kx} \mid k \in \mathbb{R}\}$: uncountably indexed.

  S_3 is a basis for $\mathrm{Span}(S_3)$.

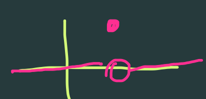
- $E = \{E_{ij} \mid 1 \le i \le m, 1 \le j \le n\}$ is a basis for $M_{m \times n}(\mathbb{R})$.

$\boxed{\text{Thm}}$ $(W, \oplus, \odot) \trianglelefteq (V, \oplus, \odot)$, assume V finite-dimensional.

- If $\dim(V) = n \in \mathbb{N}$ (fixed). Then $\dim(W) \le n$.
- $\dim(W) = n$ $\boxed{\text{iff}}$ $W = V$.

Rmk. still work for ∞-dim'l, replace n w/ $\mathrm{Card}(B)$ for B a basis of V (part 1 only)

- Ex $W = C^0(\mathbb{R}, \mathbb{R})$ & $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$. Both are ∞-dim'l but not the same!

  $f \in V \setminus W$: 

  So $W \ne V$

  W is ∞-dim'l b/c $\mathbb{P} \subseteq W$.

  $B = \{x^n \mid n \in \mathbb{N}\}$

  $\mathrm{Card}(B) = \aleph_0$

Ex $W = \{p \in \mathbb{P}^2 \mid p(3) = 2p(-1)\}$. Is a subspace of $\mathbb{P}^2$.

① $W \ne \emptyset$: $z(x) \in W$ b/c $z(3) = 0 = 2 \cdot 0 = 2 \cdot z(-1)$.

② CVA: $p_1, p_2 \in W$: •) $p_1(3) = 2 \cdot p_1(-1)$

••) $p_2(3) = 2 p_2(-1)$

$(p_1 + p_2)(3) = p_1(3) + p_2(3) = 2 p_1(-1) + 2 p_2(-1) = 2(p_1 + p_2)(-1)$. So $p_1 + p_2 \in W$.

③ CSM: exercise.

EX Cont. Find a basis. $\mathbb{P}^2$, has a basis $\{1, x, x^2\}$:

Any $p \in W \subseteq \mathbb{P}^2$: $p(x) = c_0 \cdot 1 + c_1 \cdot x + c_2 x^2$ $(c_i \in \mathbb{R})$.

$\boxed{p(3) = 2p(-1)}$

$p(3) = c_0 \cdot 1 + 3 \cdot c_1 + 9c_2$

$p(-1) = c_0 - c_1 + c_2$

$c_0 + 3c_1 + 9c_2 = 2(c_0 - c_1 + c_2) \longleftrightarrow c_0 - 5c_1 - 7c_2 = 0$ (SoE)

$\boxed{c_0 = 5c_1 + 7c_2}$

$$\left[\begin{array}{ccc|c} 1 & -5 & -7 & 0 \\ 0 & \boxed{0} & 0 & 0 \\ 0 & 0 & \boxed{0} & 0 \end{array}\right]$$

Any $p \in W$:

$p(x) = [5c_1 + 7c_2] + c_1 x + c_2 x^2$

$= c_1(5 + x) + c_2(7 + x^2)$ so $S = \{5+x, 7+x^2\}$ spans W

- NTS LI: yes (b/c different degrees!)

So: $\dim(W) = \boxed{2}$