3.5 Linear Transformations on General Vector Spaces

$T: V \to W$

Definition: A ***linear transformation:***

$$\boxed{T : (V, \oplus_V, \odot_V) \to (W, \oplus_W, \odot_W)}$$

is a function that assigns a ***unique*** member $\vec{w} \in W$ to every vector $\vec{v} \in V$, such that T satisfies for all $\vec{u}, \vec{v} \in V$ and all scalars $k \in \mathbb{R}$:

The Additivity Property :

$$\boxed{T(\vec{u} \oplus_V \vec{v}) = T(\vec{u}) \oplus_W T(\vec{v}),} \text{ and}$$

The Homogeneity Property :

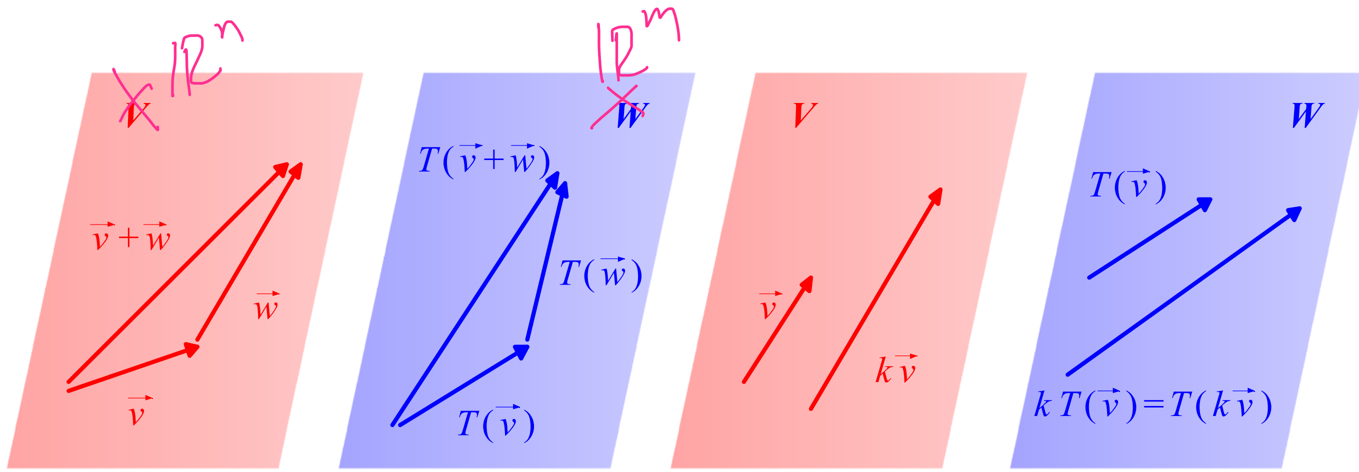
$$\boxed{T(k \odot_V \vec{v}) = k \odot_W T(\vec{v}).}$$

As usual, we write $T(\vec{v}) = \vec{w}$.

$$T : V \to W, \text{ and } T \text{ satisfies:}$$

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}), \text{ and}$$

$$T(k \cdot \vec{v}) = k \cdot T(\vec{v}),$$



The Additivity Property   The Homogeneity Property

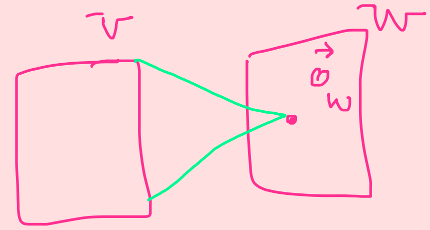
- morally correct!
- picture not always correct.

Definition: Let $(V, \oplus_V, \odot_V)$ and $(W, \oplus_W, \odot_W)$ be any two vector spaces.

The function:

$$\boxed{Z : V \to W,} \quad \text{where:}$$

$$\boxed{Z(\vec{v}) = \vec{0}_W} \text{ for all } \vec{v} \in V,$$



is a linear transformation, appropriately called the ***zero transformation***.

The function:

$$I_V : V \to V, \quad \text{where:}$$

Operator

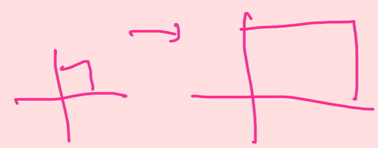
$$\boxed{I_V(\vec{v}) = \vec{v}} \text{ for all } \vec{v} \in V,$$

is called the ***identity operator*** of V.

More generally, for any scalar k, the function:

$\mathbb{R}^2$

$$S_k : V \to V, \quad \text{where:}$$



$$\boxed{S_k(\vec{v}) = k \cdot \vec{v}} \text{ for all } \vec{v} \in V,$$

$k > 0$

is called a ***scaling operator*** of V.

Evaluation Transformations

$\mathcal{F}(\mathbb{R},\mathbb{R})$ or $\mathcal{F}(I,\mathbb{R}) = V$

interval

$$E_a : F(I) \to \mathbb{R}, \quad \text{where:}$$

$$E_a(\boxed{f(x)}) = f(a),$$

If $\vec{a} = \langle a_1, a_2, \ldots, a_n \rangle \in \mathbb{R}^n$. We define:

$$\boxed{E_{\vec{a}} : V \to \mathbb{R}^n,} \quad \text{where:}$$

$$E_{\vec{a}}(\boxed{f(x)}) = \langle f(a_1), f(a_2), \ldots, f(a_n) \rangle. \quad \underline{\underline{LT}}$$

Add:

$f, g \in \mathcal{F}(\mathbb{R},\mathbb{R})$:

$$E_{\vec{a}}(\boxed{f+g}) \overset{\text{def of } E_{\vec{a}}}{=} \langle (f+g)(a_1), (f+g)(a_2), \ldots, (f+g)(a_n) \rangle$$

$$= \langle f(a_1)+g(a_1), \ldots, f(a_n)+g(a_n) \rangle \quad (\text{in } \mathbb{R}^n)$$

$$= \underbrace{\langle f(a_1), \ldots, f(a_n) \rangle} + \underbrace{\langle g(a_1), \ldots, g(a_n) \rangle} \quad (+ \text{ in } \mathbb{R}^n)$$

$$\underset{\text{def } E_{\vec{a}}}{=} E_{\vec{a}}(f) + E_{\vec{a}}(g)$$

Homogeneity

$V = \mathcal{C}^1(\mathbb{R},\mathbb{R}), \quad W = \mathcal{C}(\mathbb{R},\mathbb{R})$

Differentiation and Integration as Linear Transformations

$$\mathcal{D} : \underbrace{\mathcal{C}^1(\mathbb{R},\mathbb{R})}_{\text{all } f \text{ diff } \& \ f' \text{ is cont}} \longrightarrow \mathcal{C}(\mathbb{R},\mathbb{R})$$

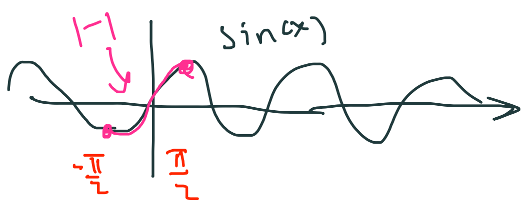
$$\boxed{\mathcal{D}(f) = f'} \quad \text{ie} \quad \mathcal{D}(f) = \frac{df}{dx} = f' = D_x(f) = \dot{f}$$

Ex $f(x) = x^3 + x + 1.$ $\quad g(x) = \sin(x)$ $\quad h(x) = e^x$

$\mathcal{D}(f) = f' = 3x^2 + 1$ $\quad \mathcal{D}(g) = \cos(x)$ $\quad \mathcal{D}(h) = e^x$

$f(x) = \sin^{-1}(x) \in V$?

$f(x) = \sin^{-1}(x) \notin V$
not defined on all of $\mathbb{R}$.



Check Linear Transformation $\quad$ Let $f, g \in V$, $k \in \mathbb{R}$;

Add $\mathcal{D}(f+g) \overset{\text{def } \mathcal{D}}{=} [f+g]' \overset{\text{Calc 1}}{=} f' + g' = \mathcal{D}(f) + \mathcal{D}(g)$

Hom $\mathcal{D}(kf) = [k \cdot f]' = k \cdot f' = k \cdot \mathcal{D}(f)$

$$\mathcal{I}_{def} : \mathcal{C}([a,b],\mathbb{R}) \longrightarrow \mathbb{R} \qquad \int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx$$

$$f \longmapsto \mathcal{I}_{def}(f) = \int_a^b f(x)\,dx$$

LT b/c Calc 1: $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

Function Spaces Preserved by the Derivative

Example: What is the smallest vector space V such that V contains the function $f(x) = x^2e^{4x}$, such that the derivative of ***every*** function in V is also a function in V?

V inside f & so $\text{Span}(f) = \{ c\, x^2 e^{4x} \mid c \in \mathbb{R} \}$.

Every $f \in V$: $f' \in V$:

$$D(f) = \frac{d}{dx}\left(f(x)\right) = [2cx]e^{4x} + (cx^2)[4e^{4x}] = 2c\,[xe^{4x}] + 4c\,(x^2e^{4x})$$

Now $\boxed{D(f) \in V} \Rightarrow \boxed{D(D(f)) \in V.}$

$$D(D(f)) = \frac{d}{dx}\left[2cx\,e^{4x} + 4cx^2\,e^{4x}\right]$$

$$= [2c]e^{4x} + (2cx)[4e^{4x}] + 4\left[2cx\,e^{4x} + 4c\,x^2e^{4x}\right]$$

$$= (2c)\,e^{4x} + (16c)\,xe^{4x} + (16c)\,x^2e^{4x} \quad \in V$$

$$D(D(\cdots D(f))) \in V$$

$$V = \text{Span}\left(\{e^{4x},\ xe^{4x},\ x^2e^{4x}\}\right).$$

(can check its a basis, but won't do it).

Arithmetic Operations on Linear Transformations

Just like linear transformations on the same Euclidean spaces, linear transformations that have the same domains and codomains can be combined using addition, subtraction and scalar multiplication:

$k \in \mathbb{R}$

Definition/Theorem: Let

$$T_1, T_2 : (V, \oplus_V, \odot_V) \to (W, \oplus_W, \odot_W)$$

be linear transformations. Then, we can define:

$$\begin{aligned} (T_1 + T_2) &: (V, \oplus_V, \odot_V) \to (W, \oplus_W, \odot_W), \\ (T_1 - T_2) &: (V, \oplus_V, \odot_V) \to (W, \oplus_W, \odot_W), \text{ and} \\ (k \cdot T_1) &: (V, \oplus_V, \odot_V) \to (W, \oplus_W, \odot_W), \end{aligned}$$

as linear transformations with actions given by:

$$\begin{aligned} (T_1 + T_2)(\vec{v}) &= T_1(\vec{v}) \oplus_W T_2(\vec{v}), \\ (T_1 - T_2)(\vec{v}) &= T_1(\vec{v}) \ominus_W T_2(\vec{v}), \text{ and} \\ (k \cdot T_1)(\vec{v}) &= k \odot_W T_1(\vec{v}). \end{aligned}$$

The vector addition, subtraction and scalar multiplication on the right side of these equations are those of the ***codomain*** W.

The Kernel and Range of a Linear Transformation

Definition/Theorem: If $T : V \to W$ is a linear transformation, we define the ***kernel*** of T as the set:

$$ker(T) = \left\{ \vec{v} \in V \,\middle|\, T(\vec{v}) = \vec{0}_W \right\}.$$

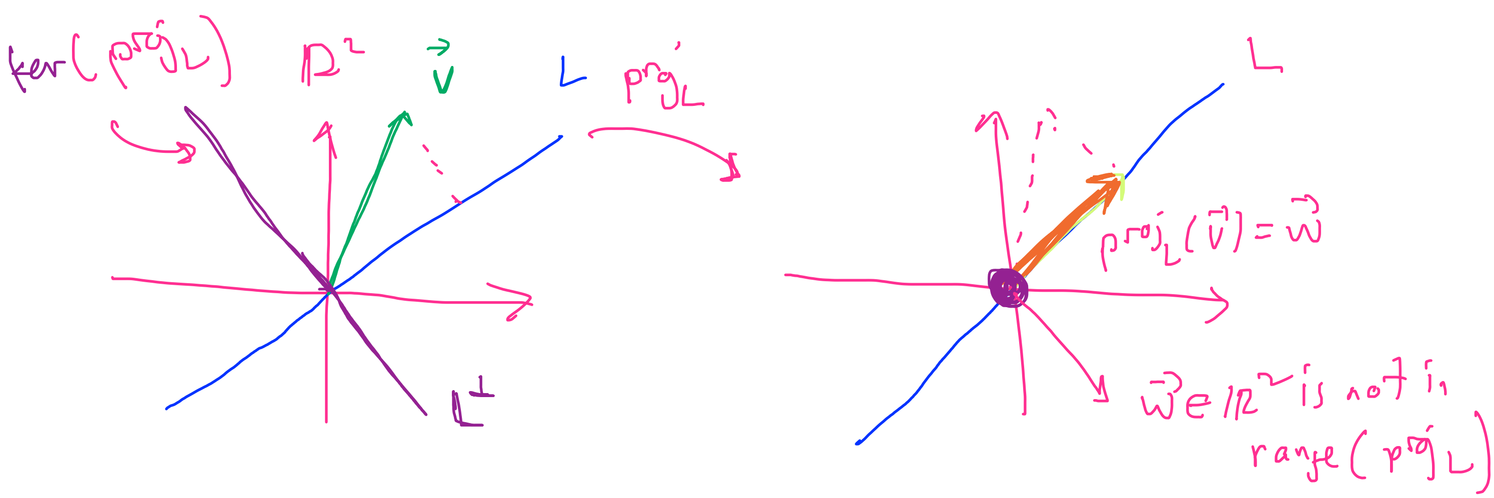
$ker(T) \trianglelefteq V$

The set $ker(T)$ is a ***subspace*** of the ***domain*** V.

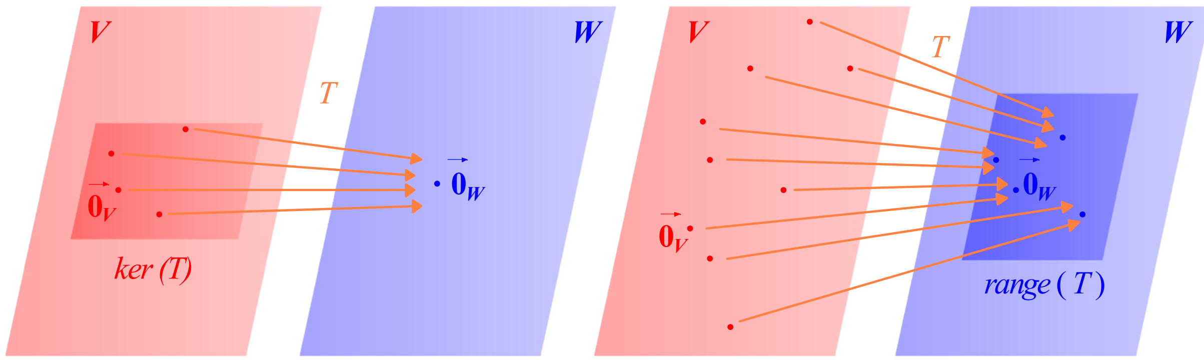
Similarly, we define the ***range*** of T as the set:

$$range(T) = \left\{ \vec{w} \in W \,\middle|\, \vec{w} = T(\vec{v}) \text{ for some } \vec{v} \in V \right\}$$

The set $range(T)$ is a ***subspace*** of the ***codomain*** W.



$range(proj_L) = L \trianglelefteq \mathbb{R}^2$



$$ker(T) = \left\{ \vec{v} \in V \mid T(\vec{v}) = \vec{0}_W \right\}$$

$$range(T) = \left\{ \vec{w} \in W \mid \vec{w} = T(\vec{v}) \text{ for some } \vec{v} \in V \right\}$$