In $\mathbb{R}^n$ $\quad \vec{v} \in \mathbb{R}^n$: $\vec{v} = \langle v_1, v_2, v_3, \dots, v_n \rangle$ $\quad$ $\mathbb{R}$ $\mathbb{R}$ $\mathbb{R}$ $\mathbb{R}$

$\boxed{T : \mathbb{R}^n \to \mathbb{R}^m}$ $\quad \vec{v} = v_1 \vec{e}_1 + v_2 \vec{e}_2 + \cdots + v_n \vec{e}_n$

$B = \{ \vec{e}_1, \vec{e}_2, \dots, \vec{e}_n \}$ basis $\mathbb{R}^n$

$\vec{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i$

3.6 Coordinate Vectors and Matrices for Linear Transformations

$T(\vec{e}_i) \in \mathbb{R}^m$ col vector w/ m rows

$\boxed{\text{Thm}}$ EQUIV LT & Mat

Key $\left[T(\vec{e}_1) \mid T(\vec{e}_2) \mid \cdots \mid T(\vec{e}_n) \right]$ col 1 col 2 $m \times n$

Definition: Let $B = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n\}$ be an ***ordered basis*** for a finite dimensional vector space V. If $\vec{v}$ is any vector in V, we know that $\vec{v}$ can be expressed ***uniquely*** as a linear combination of the vectors of B :

Thm in 3.4

$$\boxed{\vec{v} = c_1\vec{w}_1 + c_2\vec{w}_2 + \cdots + c_n\vec{w}_n.}$$

$c_1, \dots, c_n$ unique

$\vec{w}_1, \dots, \vec{w}_n$ all my basis vectors!

We call the vector $\boxed{\langle c_1, c_2, \dots, c_n \rangle}$ $\leftarrow$ MAGIC

the ***coordinate vector of $\vec{v}$ with respect to*** B, written as:

$$\boxed{\langle \vec{v} \rangle_B = \langle c_1, c_2, \dots, c_n \rangle.}$$

$\langle \vec{v} \rangle_B$ = coordinate vector representation of $\vec{v}$ wrt B

"col vector"

The $n \times 1$ matrix corresponding to $\langle \vec{v} \rangle_B$ is called the ***coordinate matrix of $\vec{v}$ with respect to*** B, written as:

$$\boxed{[\vec{v}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}}.$$

Example: $\mathbb{R}^3$

Let $B = \{\langle -1,0,1\rangle, \langle 1,1,-1\rangle, \langle 0,-1,-1\rangle\}$, assume basis.

$\vec{v} = \langle 7,-3,-2\rangle$.

Find $[\vec{v}]_B$. = LC:

$$\vec{v} = c_1\begin{bmatrix}-1\\0\\1\end{bmatrix} + c_2\begin{bmatrix}1\\1\\-1\end{bmatrix} + c_3\begin{bmatrix}0\\-1\\-1\end{bmatrix} = \begin{bmatrix}7\\-3\\-2\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}-1 & 1 & 0 & 7\\ 0 & 1 & -1 & -3\\ 1 & -1 & -1 & -2\end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{c|c}\underbrace{\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}}_{I_n} & \begin{matrix}-15\\-8\\-5\end{matrix}\end{array}\right]$$

unique!

$$\boxed{[\vec{v}]_B = \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}-15\\-8\\-5\end{bmatrix}}$$

EX $\mathbb{P}^2$, $B = \{1, x, x^2\}$, $B' = \{1+x, 30, x^2-1\}$ ordered basis

$f \in \mathbb{P}^2$: $f(x) = 1+x$ $\quad f(x) = 1\cdot(1+x) + 0\cdot 30 + 0\cdot(x^2-1)$

$f(x) = 1\cdot 1 + 1\cdot x + 0\cdot x^2$

$$[f]_{B'} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad [f]_B = \begin{bmatrix}1\\1\\0\end{bmatrix}.$$

Theorem: For any ordered basis $B = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ of an n −dimensional vector space V, the function $T : V \to \mathbb{R}^n$ given by:

$$\boxed{T(\vec{v}) = \langle \vec{v} \rangle_B}$$

is a ***linear transformation***. In particular, if $\boxed{V = \mathbb{R}^n}$ and B is a basis for $\mathbb{R}^n$, then T is in fact one-to-one and onto, i.e., an ***isomorphism of*** $\mathbb{R}^n$.

Set-up

Proof: Suppose that

$\langle \vec{u} \rangle_B = \langle c_1, c_2, \ldots, c_n \rangle$, and

$\langle \vec{v} \rangle_B = \langle d_1, d_2, \ldots, d_n \rangle$.

ie $\vec{u} \in V$: $\vec{u} = c_1\vec{w}_1 + \cdots + c_n\vec{w}_n$

$\vec{v} \in V$: $\vec{v} = d_1\vec{w}_1 + \cdots + d_n\vec{w}_n$

These mean that:

Add Prop Given $\vec{u}, \vec{v} \in V$, NTS $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

ie $\langle \vec{u}+\vec{v} \rangle_B = \langle \vec{u} \rangle_B + \langle \vec{v} \rangle_B$ ✓

In V:

$\vec{u} + \vec{v} = (c_1+d_1)\vec{w}_1 + \cdots + (c_n+d_n)\vec{w}_n$ (ass & dist)

Def of T:

$\langle \vec{u}+\vec{v} \rangle_B = \langle c_1+d_1, \ldots, c_n+d_n \rangle$ (in $\mathbb{R}^n$)

$= \langle c_1, \ldots, c_n \rangle + \langle d_1, \ldots, d_n \rangle$

$= \langle \vec{u} \rangle_B + \langle \vec{v} \rangle_B$. (def of $\langle \cdot \rangle_B$)

Homog: Proved similarly. $\square$

Coordinates for $\mathbb{P}^n$

Let $p(x) = 5x^2 - 3x + 7$. Find $[p(x)]_B$, where:

a) $B = \{1, x, x^2\}$.

b) $B' = \{x^2 - 5, x + 2, x - 1\}$.

Coordinate Vectors for $W = Span(B)$

$\text{Card}(B) = 2.$

LI

automatically basis W.

$\trianglelefteq \mathcal{F}(\mathbb{R},\mathbb{R})$

Example: Consider $\boxed{B = \{\sin(x), \cos(x)\}}$ and $\boxed{W = Span(B)}$. Find $[f(x)]_B$, for the following functions:

a) $f(x) = \underline{5}\sin(x) - \underline{8}\cos(x)$ $\rightarrow [f(x)]_B = \langle 5, -8 \rangle \neq \langle -8, 5 \rangle$

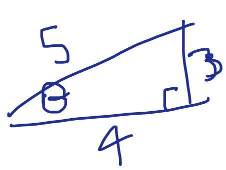
b) $f(x) = \sin\left(x + \dfrac{\pi}{4}\right) = \sin(x)\overbrace{\cos\left(\frac{\pi}{4}\right)}^{\text{const.}} + \cos(x)\overbrace{\sin\left(\frac{\pi}{4}\right)}^{\text{const.}}$

$\rightarrow [f(x)]_B = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$

c) $f(x) = \cos(x + \sin^{-1}(3/5))$

$= \cos(x)\cos(\underbrace{\sin^{-1}(\tfrac{3}{5})}_{\theta}) - \sin(x)\sin(\sin^{-1}(\tfrac{3}{5}))$

$= \frac{4}{5}\cos(x) + \left(\frac{-3}{5}\right)\sin(x)$



$[f(x)]_B = \left\langle \frac{4}{5}, \frac{-3}{5} \right\rangle$

Constructing A Matrix For T (Main Event)

↳ for two FDVSs: V, W

Definition/Theorem: Let $T : V \to W$ be a linear transformation, where $dim(V) = n$ and $dim(W) = m$. Let $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis for V, and let $B' = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m\}$ be a basis for W. ordered ordered The $\boxed{m \times n \text{ matrix } [T]_{B,B'}}$, given by:

$$[T]_{B,B'} = \Big[\underbrace{[T(\vec{v}_1)]_{B'}}_{\in \mathbb{R}^m} \Big| \underbrace{[T(\vec{v}_2)]_{B'}}_{\in \mathbb{R}^m} \Big| \cdots \Big| \underbrace{[T(\vec{v}_n)]_{B'}}_{\in \mathbb{R}^m} \Big], \quad m\times n$$

is called ***the matrix of T relative to B and B'***.

For any $\vec{v} \in V$, we can compute $T(\vec{v})$ via:

$$\boxed{[T(\vec{v})]_{B'} = [T]_{B,B'}^{*}[\vec{v}]_B.}$$

Compare w/ $T(\vec{v}) = A * \vec{v}$, $T: \mathbb{R}^n \to \mathbb{R}^m$

If $T : V \to V$ is an ***operator*** and we use the same basis B for the domain and codomain (that is, $B = B'$), we simply write $[T]_B$ instead of $[T]_{B,B}$. Otherwise, $[T]_{B,B'}$

Sketch $\vec{v} \in V$: $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$. ($B$ basis V, c_i unique)

$\forall 1 \le i \le n$ $T(\vec{v}_i) \in W$: $T(\vec{v}_i) = d_{i1}\vec{w}_1 + \cdots + d_{im}\vec{w}_m$ (B' basis W, d_i unique)

On the other hand:

MAGIC: $T(\vec{v}) = T(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n)$
$= c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \cdots + c_nT(\vec{v}_n)$

$T(\vec{v}) = d_1\vec{w}_1 + \cdots\cdots + d_m\vec{w}_m = \langle d_1, \ldots, d_m \rangle$
$= \cdots = [\text{ stuff }]\vec{w}_1 + \cdots + [\text{ stuff }]\vec{w}_m$

= finish next time! =

$$\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix}$$

How to Use the Matrix for T

ENCODE :

Given $\vec{v} \in V$, find $[\vec{v}]_B \in \mathbb{R}^n$.

MULTIPLY :

Compute the product $[T]_{B,B'}[\vec{v}]_B = [T(\vec{v})]_{B'} \in \mathbb{R}^m$. $= \begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix}$

cd vector

DECODE :

Use the coefficients of $[T(\vec{v})]_{B'}$ and the basis B' to explicitly find $T(\vec{v}) \in W$. ie

$$T(\vec{v}) = d_1\vec{w}_1 + \cdots + d_m\vec{w}_m.$$

degree 3 degree 2 differential

Example: Let $T : \mathbb{P}^3 \to \mathbb{P}^2$ be the operator given by:

$$T(p(x)) = 3p'(x) + 7xp''(x) + p(-1) \cdot x^2.$$

deg ≤ 2   deg ≤ 2   deg ≤ 2

Warm-up: Compute $T(2 + 8x - 5x^2 + 4x^3)$

168, 3, 15

$= 3[8 - 10x + 12x^2] + 7x[-10 + 24x] + [-15] \cdot x^2 = \boxed{18 - 100x + 189x^2}$

Explain why $T(p(x)) \in \mathbb{P}^2$ for any $p(x) \in \mathbb{P}^3$.

Check degree!

Prove that T is indeed a linear transformation.

Same work

$T(p_1 + p_2) = 3(p_1' + p_2') + 7x(p_1'' + p_2'') + (p_1 + p_2)(-1) \cdot x^2 = T(p_1) + T(p_2)$

Let $B = \{1, x, x^2, x^3\}$ be the standard basis for $\mathbb{P}^3$, and

$B' = \{1, x, x^2\}$ the standard basis for $\mathbb{P}^2$.

Find $[T]_{B,B'}$.

Recompute $T(2 + 8x - 5x^2 + 4x^3)$ using $[T]_{B,B'}$.

Example: Let us suppose that we are given a linear transformation $T : \mathbb{P}^2 \to \mathbb{P}^1$, with matrix:

$$[T]_{B,B'} = \begin{bmatrix} 2 & -3 & 5 \\ 4 & 1 & -2 \end{bmatrix},$$

where $B = \{x^2 + 5, x - 2, 1\}$ and $B' = \{x + 1, x - 1\}$.

$?$

$\boxed{\text{Find } T(7x^2 + 4x - 8).}$

$$[T(\underbrace{7x^2+4x-8}_{p})]_{B'} = [T]_{B,B'} * [p]_B = \begin{bmatrix} 2 & -3 & 5 \\ 4 & 1 & -2 \end{bmatrix}_{2\times 3} \begin{bmatrix} 7 \\ 4 \\ -35 \end{bmatrix}_{3\times 1}$$

$$= \boxed{\begin{bmatrix} -173 \\ 102 \end{bmatrix}_{2\times 1}}$$

$$7x^2+4x-8 = c_1(x^2+5) + c_2(x-2) + c_3(1)$$

$$= (5c_1 - 2c_2 + c_3)\cdot 1 + (c_2)\cdot x + (c_1)\cdot x^2$$

$$= (7)\cdot(x^2+5) + (4)\cdot(x-2) + (-35)\cdot 1$$

$$\begin{cases} 7 = c_1 \\ 4 = c_2 \\ -8 = 5c_1 - 2c_2 + c_3 \\ \quad = 35 - 8 + c_3 \end{cases}$$

$c_3 = -35$

$$[p]_B = \begin{bmatrix} 7 \\ 4 \\ -35 \end{bmatrix}$$

SO $\boxed{T(7x^2+4x-8) = -173(x+1) + 102(x-1)}$

Monday, 7_22_2019

Function Spaces Preserved by the Derivative

Example: Find the matrix of the derivative operator D applied to the function space:

$$V = Span(\{x^2e^{4x}, xe^{4x}, e^{4x}\})$$

Revisiting Projections

Skip -- see book.

Example: Suppose that Π is the plane with equation: $5x + 2y - 6z = 0$.

Find $[proj_{\Pi}]$.