3.7 1-1, Onto LTs & Composition of LTs

$\boxed{\text{Thm}}$ Dimension Theorem (VS version)

Let $T: V \to W$ LT

let V fdvs, $\dim(V) = n$.

Then

$$\boxed{\dim(V) = \text{rank}(T) + \text{nullity}(T)}$$

Rmk ① Only need V to be finite dimensional

② Don't need to assume T is 1-1 or onto!

Proof Since V is fdvs, & $\ker(T) \trianglelefteq V$, $\ker(T)$ is fdvs.

Assume $\text{nullity}(T) = \dim(\ker(T)) = k$, with $k \leq n$.

Case $k > 0$. [Then T is not 1-1 (Note).]

Since every vector space has a basis, $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ (LI + Span), for $\ker(T)$.

Case $k = 0$. Then T is 1-1, so $\ker(T) = \{\vec{0}_V\}$.

Let $S = \emptyset$.

By (~~Basis~~) extension theorem: can extend S to a basis for V.

Write S as follows:

$$S = \{\underbrace{\vec{v}_1, \ldots, \vec{v}_k}_{\text{basis for } \ker(T)}, \vec{v}_{k+1}, \ldots, \vec{v}_n\} \quad \text{basis for } V.$$

Thus, for any $\vec{v} \in V$: can write uniquely as:

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k + c_{k+1}\vec{v}_{k+1} + \cdots + c_n\vec{v}_n$$

Now use that T is a LT:

$$T(\vec{v}) = T(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k + c_{k+1}\vec{v}_{k+1} + \cdots + c_n\vec{v}_n)$$

$$= \left[c_1\underbrace{T(\vec{v}_1)}_{\vec{0}_W} + \cdots + c_k\underbrace{T(\vec{v}_k)}_{\vec{0}_W}\right] + c_{k+1}T(\vec{v}_{k+1}) + \cdots + c_nT(\vec{v}_n)$$

$$= \left[\vec{0}_W + \cdots + \vec{0}_W\right] + c_{k+1}T(\vec{v}_{k+1}) + \cdots + c_nT(\vec{v}_n).$$

$$= c_{k+1}\underbrace{T(\vec{v}_{k+1})} + \cdots + c_n\underbrace{T(\vec{v}_n)}.$$

Hopefully $B' = \{T(\vec{v}_{k+1}), \ldots, T(\vec{v}_n)\}$ is a basis for $Range(T) \leq W$.

We already know spanning! $T(\vec{v}) \in Span(\{\vec{v}_{k+1}, \ldots, \vec{v}_n\})$.

NTS B' is LI!

Set-up DTE Let $r_{k+1}, \ldots, r_n \in \mathbb{R}$:

$\boxed{\text{DTE}}$ $\quad r_{k+1}T(\vec{v}_{k+1}) + \cdots + r_nT(\vec{v}_n) = \vec{0}_W \qquad$ NTS $r_{k+1} = \cdots = r_n = 0$.

Use linearity of T:

$$T(r_{k+1}\vec{v}_{k+1} + \cdots + r_n\vec{v}_n) = \vec{0}_W.$$

So: $r_{k+1}\vec{v}_{k+1} + \cdots + r_n\vec{v}_n \in \ker(T) = Span(\{\vec{v}_1, \ldots, \vec{v}_k\})$.

But $S = \{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ basis for V

$\hookrightarrow$ LI

Immediately get: $r_{k+1} = 0,\ r_{k+2} = 0, \dots, r_n = 0$!

- So rank$(T) = \dim(\text{Range}(T)) = \text{Card}(\text{basis for Range}(T))$

$$= \text{Card}(B')$$

$$= n-k$$

- This proves the Dimension Theorem since

$$k = \dim(\ker(T)) = \text{nullity}(T)$$
$$n = \dim(V)$$

So $\text{rank}(T) = \dim(V) - \text{nullity}(T)$. $\square$

Consequence of Dim Thm:

Example
- $T: \mathbb{R}^5 \to \mathbb{P}^2$ is LT. What can you conclude?

  $\hookrightarrow$ $\dim V > \dim W \to$ not 1-1

- $T: \mathbb{P}^7 \to \mathbb{R}^{10}$ is LT.

  $\hookrightarrow$ $\dim(V) = 8 \qquad \dim(W) = 10$

Example Function Spaces preserved by Derivative

$D: V \to V$ where $V = \text{Span}(\underbrace{\{x^2 \cdot e^{4x},\ x \cdot e^{4x},\ e^{4x}\}}_{\text{ordered basis } B})$.

Know from last section:

$$[D]_B = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 4 & 0 \\ 0 & 1 & 4 \end{bmatrix}_B$$

a) Find the matrix representation for the second derivative $D^2(f) = f''$

$D^2 = D \circ D$

b) Find second derivative of $f(x) = 7x^2e^{4x} - 2xe^{4x} + 8e^{4x}$.

Sol (a) $[D^2]_B \overset{\text{Theorem}}{=} [D]_B * [D]_B$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 2 & 4 & 0 \\ 0 & 1 & 4 \end{bmatrix} * \begin{bmatrix} 4 & 0 & 0 \\ 2 & 4 & 0 \\ 0 & 1 & 4 \end{bmatrix} = \boxed{\begin{bmatrix} 16 & 0 & 0 \\ 16 & 16 & 0 \\ 2 & 8 & 16 \end{bmatrix}}$$

(b) $[f]_B = \begin{bmatrix} 7 \\ -2 \\ 8 \end{bmatrix}_B$. $[D^2(f)]_B = [D^2] * [f]_B$

$$= \begin{bmatrix} 16 & 0 & 0 \\ 16 & 16 & 0 \\ 2 & 8 & 16 \end{bmatrix}_B \begin{bmatrix} 7 \\ -2 \\ 8 \end{bmatrix}_B = \begin{bmatrix} 112 \\ 80 \\ 126 \end{bmatrix}_B$$

$$D^2(f) = f''(x) = \boxed{112x^2e^{4x} + 80xe^{4x} + 126e^{4x}}$$