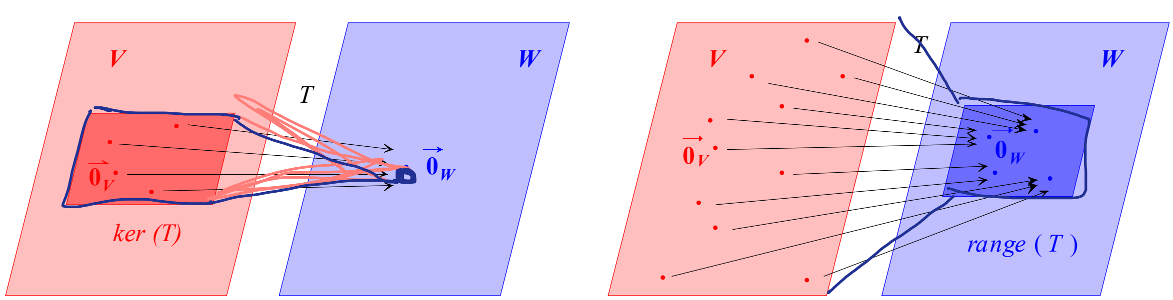
3.7 One-to-One and Onto Linear Transformations ; Compositions of Linear Transformations

Review:



$$ker(T) = \left\{ \vec{v} \in V \mid T(\vec{v}) = \vec{0}_W \right\}$$

$$range(T) = \left\{ \vec{w} \in W \mid \vec{w} = T(\vec{v}) \text{ for some } \vec{v} \in V \right\}$$

One-to-One Transformations

$\mathbb{R}^n$ $\mathbb{R}^m$

Definition: We say that a linear transformation $T : V \to W$ is ***one-to-one*** or ***injective*** if the image of different vectors from the domain are different vectors from the codomain:

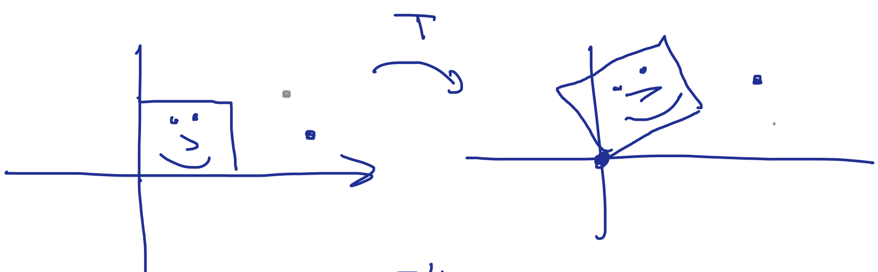
$\vec{v}_1, \vec{v}_2 \in V$

$$\text{If } \vec{v}_1 \neq \vec{v}_2 \text{ then } T(\vec{v}_1) \neq T(\vec{v}_2).$$

[exactly same as before]

We again say that T is an ~~***injection***~~ or an ~~***embedding***~~.

Ex $T: \mathbb{R}^2 \to \mathbb{R}^2$, $[T] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$



is 1-1.

$\theta = \pi/6$

Theorem: A linear transformation $T : V \to W$ is ***one-to-one*** if and only if the only way two vectors from the domain have the same image in the codomain is for them to be the same vector to begin with:

[contrapositive] ☆ If $T(\vec{v}_1) = T(\vec{v}_2)$ then $\vec{v}_1 = \vec{v}_2$. (Useful one)

In other words, the only solution to $T(\vec{v}_1) = T(\vec{v}_2)$ is $\vec{v}_1 = \vec{v}_2$.

Theorem: A linear transformation $T : V \to W$ is one-to-one ***if and only if*** $ker(T) = \left\{\vec{0}_V\right\}$.

Same exact proof as in $V=\mathbb{R}^n$, $W=\mathbb{R}^m$.

($\Leftarrow$) Assume $\ker(T) = \{\vec{0}_V\}$. WTS T is 1-1.

Let $\vec{v}_1, \vec{v}_2 \in V$. Let $T(\vec{v}_1) = T(\vec{v}_2)$. Then:

$\vec{0}_W = T(\vec{v}_1) - T(\vec{v}_2)$ (by assumption)

$= T(\vec{v}_1 - \vec{v}_2)$ (both properties: Homog. & Add Prop)

So $\vec{v}_1 - \vec{v}_2 \in \ker(T) = \{\vec{0}_V\}$. So: $\vec{v}_1 - \vec{v}_2 = \vec{0}_V$. So: $\vec{v}_1 = \vec{v}_2$. $\square$

($\Rightarrow$) exercise. $\square$

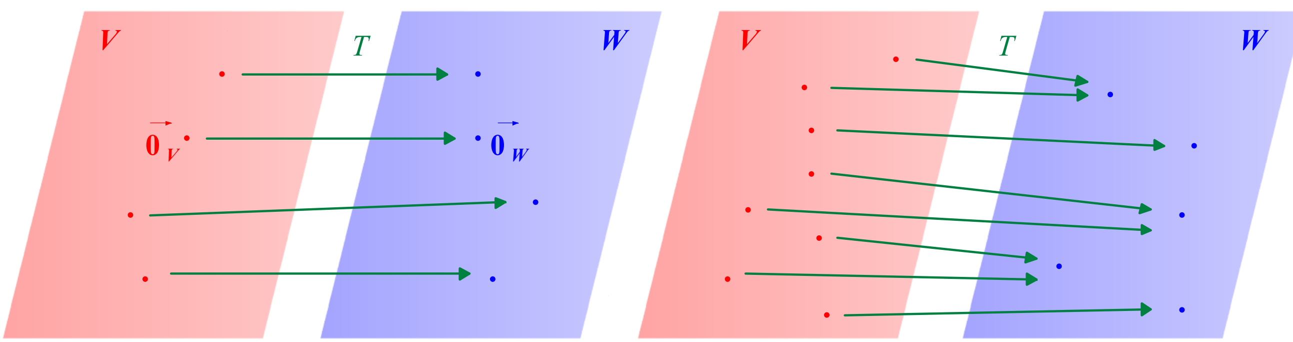
Onto Transformations

$\mathbb{R}^n$ $\mathbb{R}^m$

Definition: We say that a linear transformation $T : V \to W$ is ***onto*** or ***surjective*** if the range of T is ***all*** of W :

def $$\boxed{range(T) = W.}$$

Since $rank(T) = dim(range(T))$, we can also say that T is ***onto*** ***if and only if*** $\boxed{rank(T) = dim(W)}$, in the case when W is ***finite dimensional***.



one-to-one $\Leftrightarrow$

$$ker(T) = \left\{\vec{0}_V\right\}$$

onto $\Leftrightarrow$

$$range(T) = W$$

Finding the Kernel and Range Using $[T]_{B,B'}$

matrix GJR

The information provided by $[T]_{B,B'}$ and its rref simply needs to be ***decoded*** with respect to the appropriate basis: we use B for $ker(T)$ and B' for $range(T)$.

Theorem: Suppose that $T : V \to W$ is a linear transformation, with $dim(V) = n$ and $dim(W) = m$, both finite-dimensional vector spaces. Let $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis for V, and let $B' = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m\}$ be a basis for W. Let us construct the $m \times n$ matrix $[T]_{B,B'}$ as we did in the previous Section, and let R be the rref of $[T]_{B,B'}$. Suppose that:

$$\{\boxed{\vec{z}_1}, \boxed{\vec{z}_2}, \ldots, \boxed{\vec{z}_k}\} \subset \mathbb{R}^n \quad = NS(R) = NS(A)$$

is the basis that we obtain for $nullspace\left([T]_{B,B'}\right)$ using R, as we did in Chapter 2.

By the Uniqueness of Representation Property, we know that there exists $\vec{u}_i \in V$ so that $\langle \vec{u}_i \rangle_B = \vec{z}_i$ for every $i = 1 \ldots k$.

We conclude that the set $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\} \subset V$ is a ***basis*** for $ker(T)$.

As usual, if there are no free variables in R, then $nullspace\left([T]_{B,B'}\right) = \left\{\vec{\mathbf{0}}_n\right\}$,

and consequently, $ker(T) = \left\{\vec{\mathbf{0}}_V\right\}$. 1-1

So: $ker(T)$ has basis $\{\vec{u}_1, \ldots, \vec{u}_k\} \subseteq V$.

free variables $= dim(ker(T)) = nullity(T)$.

from [T]

Similarly, the set of original columns:

$$\{\vec{c}_{i_1}, \vec{c}_{i_2}, \ldots, \vec{c}_{i_r}\} \subset \mathbb{R}^m$$

from $[T]_{B,B'}$ corresponding to the leading 1's of R form a basis for $columnspace\left([T]_{B,B'}\right)$ as we found in Chapter 2, and there exists $\vec{d}_j \in W$ so that $\left\langle \vec{d}_j \right\rangle_{B'} = \vec{c}_{i_j}$ for every $j = 1 \ldots r$.

We conclude that the set $\left\{\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_r\right\} \subset W$ is a ***basis*** for $range(T)$.

If T is the zero transformation, then $range(T) = \left\{\vec{\mathbf{0}}_W\right\}$.

basis

$\{\vec{C}_{i_1}, \ldots, \vec{C}_{i_r}\}$ col in [T] w/ leading 1

(decode to W (b/c range(T) ⊆ W))

No proof.

The Dimension Theorem for Abstract Vector Spaces

Key Result in LA!!! ...

Theorem — The Dimension Theorem: → general: doesn't need to be 1-1 or onto!

Let $T: V \to W$ be a linear transformation, and suppose that V ***is finite dimensional*** with $dim(V) = n$. Then, both $ker(T)$ and $range(T)$ are finite dimensional, and we can define:

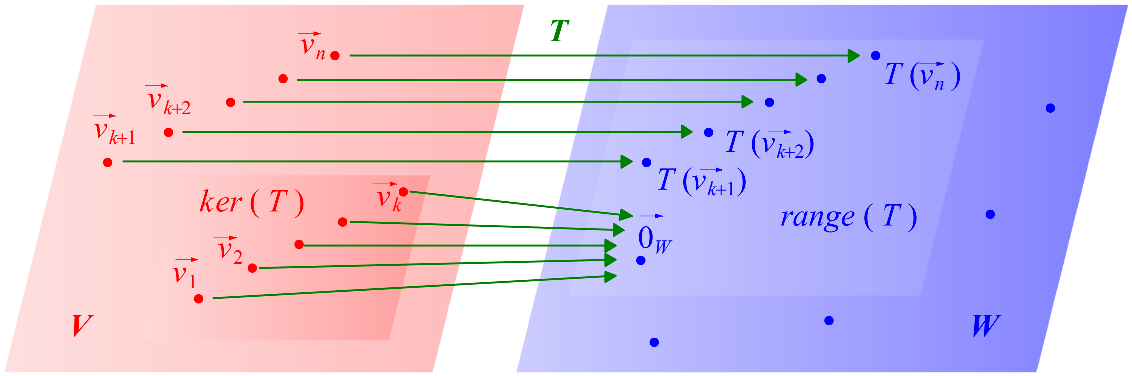
$$rank(T) = dim(range(T)), \text{ and}$$ card(B_1)

$$nullity(T) = dim(ker(T)).$$ card(B_2)

Furthermore, these quantities are related by the equation:

$$rank(T) + nullity(T)$$
$$= n = dim(V) = dim\left(domain \text{ of } T\right).$$

$$\boxed{dim(V) = rank(T) + nullity(T)}$$



Idea of the Proof

Let's say $nullity(T) = k$.

$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ a basis for $ker(T)$

Use the ***Extension Theorem***:

$B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k, \vec{v}_{k+1}, \vec{v}_{k+2}, \ldots, \vec{v}_n\}$ a basis for V.

Examine $range(T)$:

$$\begin{aligned}\vec{v} = {} & c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k + \\ & c_{k+1}\vec{v}_{k+1} + c_{k+2}\vec{v}_{k+2} + \cdots + c_n\vec{v}_n.\end{aligned}$$

$$\begin{aligned}T(\vec{v}) = {} & c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + \cdots + c_k T(\vec{v}_k) + \\ & c_{k+1} T(\vec{v}_{k+1}) + c_{k+2} T(\vec{v}_{k+2}) + \cdots + c_n T(\vec{v}_n) \\ = {} & c_{k+1} T(\vec{v}_{k+1}) + c_{k+2} T(\vec{v}_{k+2}) + \cdots + c_n T(\vec{v}_n)\end{aligned}$$

This tells us that every vector in $range(T)$ is . . . .

Show that $\{T(\vec{v}_{k+1}), T(\vec{v}_{k+2}), \ldots, T(\vec{v}_n)\}$ is linearly independent:

$$d_{k+1}T(\vec{v}_{k+1}) + d_{k+2}T(\vec{v}_{k+2}) + \cdots + d_nT(\vec{v}_n) = \vec{0}_W.$$

$$T(d_{k+1}\vec{v}_{k+1} + d_{k+2}\vec{v}_{k+2} + \cdots + d_n\vec{v}_n) = \vec{0}_W.$$

Conclusion: the vector inside the parenthesis is in . . .

$$\begin{aligned} &d_{k+1}\vec{v}_{k+1} + d_{k+2}\vec{v}_{k+2} + \cdots + d_n\vec{v}_n \\ = \; &d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_k\vec{v}_k, \end{aligned}$$

$$\begin{gathered} -d_1\vec{v}_1 - d_2\vec{v}_2 - \cdots - d_k\vec{v}_k + \\ d_{k+1}\vec{v}_{k+1} + d_{k+2}\vec{v}_{k+2} + \cdots + d_n\vec{v}_n = \vec{0}_W \end{gathered}$$

Conclusion:

Comparing Dimensions

Theorem: Suppose $T : V \to W$ is a linear transformation of finite dimensional vector spaces. Then:

a) if $dim(V) < dim(W)$, then T ***cannot*** be onto.

b) if $dim(V) > dim(W)$, then T ***cannot*** be one-to-one.

Compositions of Linear Transformations

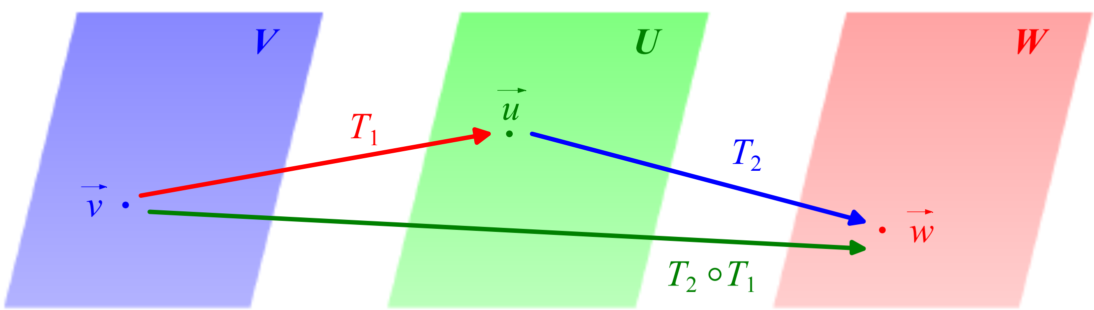
eg: $\mathbb{R}^n$ $\mathbb{R}^k$ $\mathbb{R}^k$ $\mathbb{R}^m$

Definition/Theorem: Suppose $T_1 : V \to U$, and $T_2 : U \to W$ are linear transformations. The ***composition:***

$$\boxed{T_2 \circ T_1 : V \to W}$$

is again a linear transformation, with action given as follows: Suppose $\vec{v} \in V$, $T_1(\vec{v}) = \vec{u} \in U$, and $T_2(\vec{u}) = \vec{w} \in W$. Then:

$$\boxed{(T_2 \circ T_1)(\vec{v}) = T_2(T_1(\vec{v}))} = T_2(\vec{u}) = \vec{w}.$$



The Composition of

Two Linear Transformations

Note these can all be ∞-dim'l

NTS • Add Prop: ✓

• Homogeneity Prop: ✓ Do at home.

The Matrix of a Composition

Theorem: Let $\underbrace{T_1 : V \to U}$ and $\underbrace{T_2 : U \to W}$ be linear transformations of finite dimensional vector spaces. Let B be a basis for V, $B^{/}$ a basis for U, and $B^{//}$ a basis for W. Then:

n k k m

$$[T_2 \circ T_1]_{B,B^{//}} = [T_2]_{B^{/},B^{//}} * [T_1]_{B,B^{/}}.$$

Proof:

Set-up.

$B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ basis V

$B^{/} = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ basis U

$B^{//} = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m\}$ basis W

- $[T_2 \circ T_1]_{B,B''} = Z_{m\times n}$ (by Thm in §3.6)

$\| [\text{Thm}]$

$\left[[T_2 \circ T_1(\vec{v}_1)]_{B''} \,\Big|\, \cdots\cdots \,\Big|\, [T_2 \circ T_1(\vec{v}_n)]_{B''} \right]$

- $[T_1]_{B,B'} = \left[[T_1(\vec{v}_1)]_{B'} \,\Big|\, \cdots \,\Big|\, [T_1(\vec{v}_n)]_{B'} \right] = Y_{k\times n}$

- $[T_2]_{B',B''} = \left[[T_2(\vec{u}_1)]_{B''} \,\Big|\, \cdots \,\Big|\, [T_2(\vec{u}_k)]_{B''} \right] = X_{m\times k}$

SO WTS $\boxed{Z_{m\times n} = X_{m\times k} * Y_{k\times n}}$

- First column of Z : $[T_2 \circ T_1(\vec{v}_1)]_{B''} = [T_2(T_1(\vec{v}))]_{B''}$

$= [T_2]_{B',B''} * [T_1(\vec{v}_1)]_{B'}$

$= X * \left([T_1]_{B,B'} * [\vec{v}_1]_B\right) = X * Y * [\vec{v}_1]$

• First Column of $X * Y$: $X * (1^{st} \text{ cl } Y) = X * [T_1(\vec{v}_1)]_{B'}$

$= X * \left([T_1]_{B,B'} * [\vec{v}_1]_B \right)$

• So works for all columns too!

$\square$

$$[T_2 \circ T_1]_{B,B''}$$

$$= Z$$

$$= \Big[\ [(T_2 \circ T_1)(\vec{v}_1)]_{B''}\ \Big|\ [(T_2 \circ T_1)(\vec{v}_2)]_{B''}\ \Big|\ \cdots\ [(T_2 \circ T_1)(\vec{v}_n)]_{B''}\ \Big]$$

$$[T_2]_{B',B''}$$

$$= X$$

$$= \Big[\ [T_2(\vec{u}_1)]_{B''}\ \Big|\ [T_2(\vec{u}_2)]_{B''}\ \Big|\cdots\Big|\ [T_2(\vec{u}_k)]_{B''}\ \Big]$$

$$[T_1]_{B,B'}$$

$$= Y$$

$$= \Big[\ [T_1(\vec{v}_1)]_{B'}\ \Big|\ [T_1(\vec{v}_2)]_{B'}\ \Big|\cdots\Big|\ [T_1(\vec{v}_n)]_{B'}\ \Big]$$

Function Spaces Preserved by the Derivative

Example: Find the matrix of the ***second derivative***, D^2, applied to the function space:

$$V = Span(\{x^2e^{4x}, xe^{4x}, e^{4x}\})$$