Def $\mathcal{L}(V, W) = \{ T : V \to W \mid T \text{ is a LT} \}$

= set of all linear transformations from V to W

Thm $\mathcal{L}(V, W)$ is a vectorspace!

When V & W are finite-dimensional, so is $\mathcal{L}(V, W)$.

↳ In this case, the dimension of $\mathcal{L}(V, W)$ is

$$\dim(V) \cdot \dim(W) = n \cdot m = \dim(\mathrm{Mat}_{m \times n}).$$

Pf

- VS ✓
- Now, V^n & W^m finite-dimensional.
- $\Phi : \mathcal{L}(V, W) \to \mathrm{Mat}_{m \times n}$

  $T \longmapsto \Phi(T) = [T]_{B, B'}$
- Φ is isomorphism then $\mathcal{L}(V, W)$ & $\mathrm{Mat}_{m \times n}$ have same dimension!
- Φ LT, 1-1, onto } Φ isomorphism.
- Φ LT ✓

- Φ is 1-1

Let $T \in \mathcal{L}(V, W)$ and $T \in \operatorname{Ker}(\Phi)$.

Then $\Phi(T) = O_{m \times n}$ (the $m \times n$ zero matrix)

So, $[T]_{B,B'} = O_{m \times n}$.

Let $B = \{\vec{v}_1, \ldots, \vec{v}_n\} \subseteq V$ is a basis.

Then for each $i = 1, 2, \ldots, n$:

$$[T(\vec{v}_i)]_{B'} = [T]_{B,B'} * [\vec{v}_i]_B \quad \text{(Thm on matrix Rep)}$$

↳ is the i^{th} column of $[T]_{B,B'} = O_{m \times n}$

so $[T(\vec{v}_i)] = \vec{O}_m$ true for each i.

On the other hand:

$$[T] = [\, T(\vec{v}_1) \,|\, T(\vec{v}_2) \,|\, \cdots \,|\, T(\vec{v}_n) \,] = O_{m \times n}$$

So for <u>any</u> $\vec{v} \in V$: $\exists c_1, \ldots, c_n \in \mathbb{R}$:

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

$$[T(\vec{v})]_{B'} = c_1 [T(v_1)]_{B'} + \cdots + c_n [T(\vec{v}_n)]_{B'}$$

(by Linearity of T)

$$= \vec{O}_m.$$

$T(\vec{v}) = 0 \cdot \vec{w}_1 + 0\vec{w}_2 + \cdots + 0\vec{w}_m$ in $B' = \{\vec{w}_1, \ldots, \vec{w}_m\}$

so $\forall \vec{v}$! So T is the zero transformation from V to W.

So $\ker(\Phi) = \{0_{V,W}\}$. So Φ is 1-1.

- **Φ is onto** For any matrix $A \in \text{Mat}_{m\times n}$ there's a LT $T \in \mathcal{L}(V,W)$ so that

$$\Phi(T) = A, \text{ ie } [T]_{B,B'} = A.$$

Write $A = (a_{ij})_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & \ddots & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix}$

Define $T: V \to W$ as follows:

define for basis vectors $\vec{v}_1, \dots, \vec{v}_n$:

$$T(\vec{v}_j) = \sum_{i=1}^{m} a_{ij}\vec{w}_i = a_{1j}\vec{w}_1 + a_{2j}\vec{w}_2 + \cdots + a_{mj}\vec{w}_m$$

(check this is a LT).

$$[T(\vec{v}_j)]_{B'} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} = j^{th} \text{ column of } A \text{ !!!}$$

Recall

$$[T]_{B,B'} = \left[[T(\vec{v}_1)]_{B'} \,\middle|\, \cdots \,\middle|\, [T(\vec{v}_n)]_{B'} \right] = A$$

So $[T] = A$, ie $\Phi(T) = A$. So Φ is onto. $\square$