Section 3.8 Isomorphisms

when using abstract vector spaces.

Definition: If V and W are vector spaces, we say that a linear transformation $T : V \to W$ is an ***isomorphism*** if T is both ***one-to-one*** and ***onto***. We also say that T is ***invertible***, T is ***bijective***, and that V and W are ***isomorphic*** to each other.

If $V = W$, an isomorphism $T : V \to V$ is also called an ***automorphism*** or self-isomorphism.

The Usual Dimension Requirement

Previously: $T: \mathbb{R}^n \to \mathbb{R}^m$ invertible iff $n=m$

1-1 & onto

Theorem: Suppose $T : V \to W$ is an isomorphism of finite dimensional vector spaces. Then $\boxed{dim(V) = dim(W)}$.

Pf If $dim(V) < dim(W)$ then T isn't onto. So this can't happen. Then $dim(V) \geq dim(W)$. If $dim(V) > dim(W)$ then T isn't 1-1 since $dim(ker(T)) = dim(V) - rank(T) > 0$. Can't have that either so $dim(V) = dim(W)$. □

A partial converse is also true (see the Exercises).

Surprising!!!

☆

Theorem: If V and W are *finite dimensional* vector spaces and $dim(V) = dim(W)$, $\boxed{\text{then}}$ ***there exists*** an *isomorphism* $T : V \to W$.

LT & 1-1 & onto

Corollary If V any vector space & $dim(V)=n$ then $\exists$ $T: V \to \mathbb{R}^n$ that's an isomorphism!

Put them together:

Theorem: Two vector spaces V and W are *isomorphic* to each other ***if and only if:***

$$dim(V) = dim(W).$$

The Existence of the Inverse

$$: V \xrightarrow{T} W, \quad W \xrightarrow{S} V$$

Definition/Theorem: A linear transformation $T : V \to W$ is an ***isomorphism*** of vector spaces ***if and only if*** there exists another linear transformation:

$$S = T^{-1} : W \to V,$$

$S \circ T = Id_V, \quad T \circ S = Id_W$

called the ***inverse*** of T, which is ***also*** an ***isomorphism***, such that if $\vec{v} \in V$ and $T(\vec{v}) = \vec{w} \in W$, then $T^{-1}(\vec{w}) = \vec{v}$, and thus:

$$\boxed{(T^{-1} \circ T)(\vec{v}) = \vec{v}} \text{ and } \boxed{(T \circ T^{-1})(\vec{w}) = \vec{w}.}$$

In other words, T^{-1} is also a ***one-to-one*** and ***onto*** linear transformation.

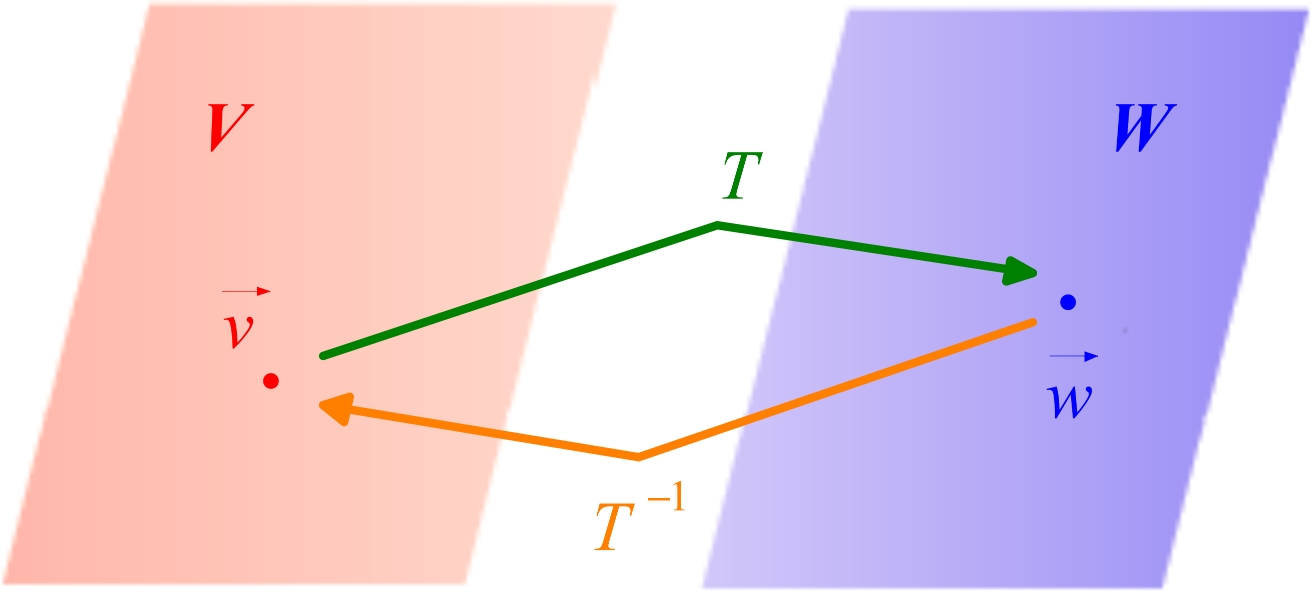
Furthermore, T^{-1} is ***unique***, and T and T^{-1} possess the ***cancellation properties:***

$$T^{-1} \circ T = I_V \quad \text{and} \quad T \circ T^{-1} = I_W,$$

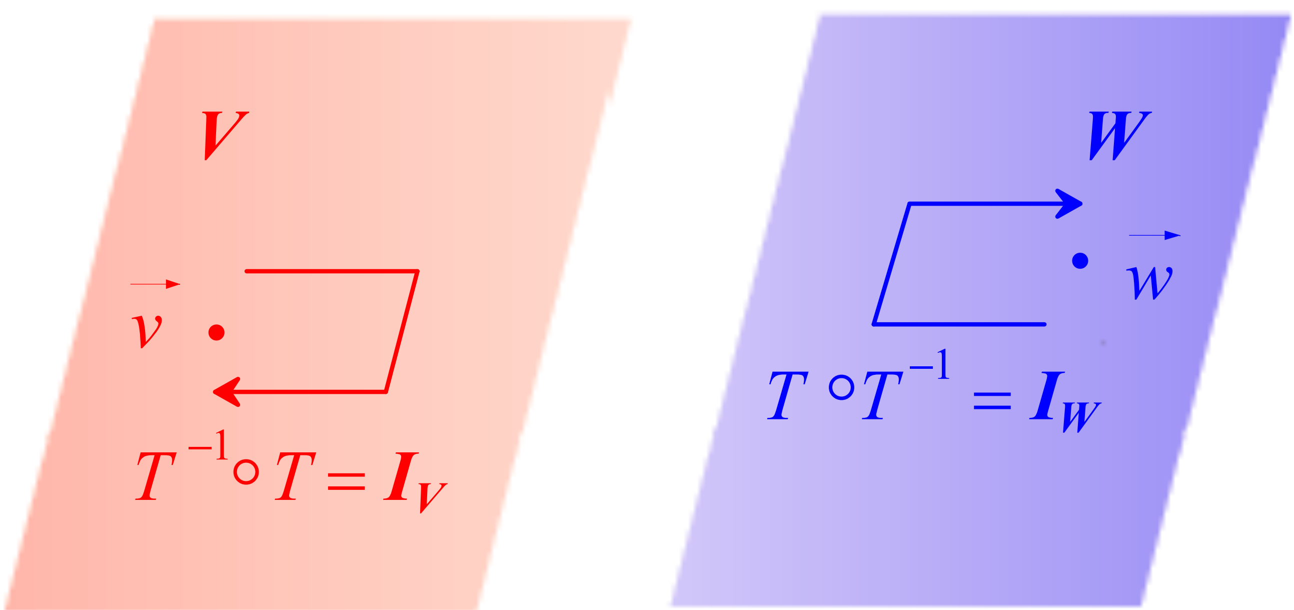
where I_V and I_W are the ***identity*** operators on V and W, respectively.

In particular, if T is an ***automorphism***, we get: $T^{-1} \circ T = I_V = T \circ T^{-1}$.

Pf Exactly same as $\mathbb{R}^n$! $\square$



The Composition of T with T^{-1}



$\boxed{T^{-1} \circ T = I_V}$ and $\boxed{T \circ T^{-1} = I_W}$

The Matrix of the Inverse

Theorem: Suppose $T : V \to W$ is an isomorphism of ***finite dimensional*** vector spaces. By the previous Theorems, we know that $dim(V) = dim(W) = n$, say, and there exists $T^{-1} : W \to V$ such that $T^{-1} \circ T = I_V$ and $T \circ T^{-1} = I_W$. If B is a ***basis*** for V and B' is a ***basis*** for W, then $[T]_{B,B'}$ is an ***invertible*** $n \times n$ matrix, and:

$$\boxed{[T^{-1}]_{B',B} = [T]^{-1}_{B,B'}.}$$

In particular, if $T : V \to V$ is an ***automorphism***, then:

$$[T^{-1}]_B = [T]^{-1}_B.$$

Pf Assume T is an isomorphism. Previous theorem says $T^{-1}: W \to V$ exists and is also an isomorphism. Moreover:

$$T^{-1} \circ T = Id_V \quad \& \quad T \circ T^{-1} = Id_W$$

Big result about composition & matrices tells us:

$$[T^{-1} \circ T]_{B,B} = [Id_V]_{B,B}$$

$$[T^{-1}]_{B',B} * [T]_{B,B'} = [Id_V]_B = I_n$$

& similarly:

$$[T]_{B,B'} * [T^{-1}]_{B',B} = [Id_W]_{B'} = I_n$$

all happening in Euclidean spaces $\Leftrightarrow$ matrices!

Since Inverses are unique, we get $[T^{-1}]_{B',B} = [T]^{-1}_{B,B'}$ $\square$

Application: Solving for Derivatives and Antiderivatives

Application: Solving Ordinary Differential Equations

$$c_n y^{(n)} + \cdots + c_2 y^{(2)} + c_1 y' + c_0 y = g(x)$$

Use $g(x)$ to "guess" an appropriate function space:

$$W = Span(\{g_1(x), g_2(x), \ldots g_k(x)\})$$

arising from $g(x)$ and its derivatives.

Application: Curve Fitting

We know from basic algebra that two distinct points determine a unique line.

Similarly, three non-collinear points will determine a unique parabolic function $p(x) = ax^2 + bx + c$.

If the points are collinear, we get a "degenerate" quadratic $p(x) = bx + c$ or a constant polynomial $p(x) = c$, but notice that all these polynomials are members of $\mathbb{P}^2$.

Continuing with this analogy, ***four points*** with ***distinct x-coordinates*** will determine a unique polynomial of ***at most third degree***, in other words, a member of $\mathbb{P}^3$, and so on.

(The fact that the transformation T that we produce is invertible will be seen in the Exercises of Section 5.3).