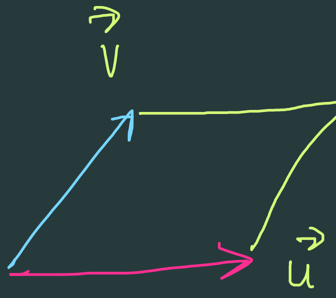
5.3 Properties of Determinants and Cofactor Expansion

$$\boxed{\det : \mathrm{Mat}_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R}}$$ (not a linear map!)

want to build:

$= ad-bc$

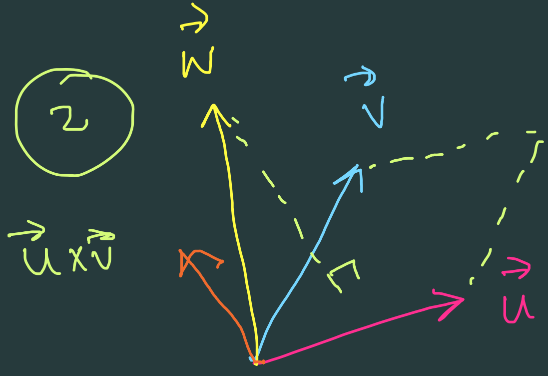
① In 2x2 case: $\det(A)$ = signed ~~volume~~ area of parallelogram spanned two columns A



$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\boxed{\text{Thm}}$ $\det(A) \neq 0$ iff $ad-bc \neq 0$
iff $\vec{u}$ & $\vec{v}$ <u>not</u> parallel.

②



$\det(A)$ = signed volume parallelopiped $\vec{u}, \vec{v}, \vec{w}$

signed vol $= \|\vec{u}\times\vec{v}\| \cdot$ "height"

$= \|\vec{u}\times\vec{v}\| \cdot \mathrm{comp}_{\vec{u}\times\vec{v}}(\vec{w})$

"triple scalar product"

$$= \cancel{\|\vec{u}\times\vec{v}\|} \cdot \frac{(\vec{w}\cdot(\vec{u}\times\vec{v}))}{\cancel{\|\vec{u}\times\vec{v}\|}} = \vec{w}\cdot(\vec{u}\times\vec{v})$$

$$= \begin{vmatrix} \multicolumn{3}{c}{\boxed{w_1 \quad w_2 \quad w_3}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = w_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - w_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}$$

$$+ w_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

Generalize to $\mathbb{R}^n$?

Let $A \in \mathrm{Mat}_{n\times n}$

Thm

- $\det(A) = 0$ $\boxed{\text{iff}}$ A is not invertible.
- $\det(A) \neq 0$ $\boxed{\text{iff}}$ A is invertible.
- $\det(A * B) = \det(A) * \det(B)$.

5.1 Summary

- def $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(A) = ad - bc$
- def $A = \begin{bmatrix} \multicolumn{3}{c}{\boxed{a_{11} \quad a_{12} \quad a_{13}}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $\det(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

- **Permutations** "rearrangements"

- $\sigma : \{1,2,\dots,n\} \to \{1,2,\dots,n\}$ bijection (1-1 & onto)
  ↳ σ^{-1} exists $\sigma^{-1} \circ \sigma = id$

- $\operatorname{sign}(\sigma) = \pm 1 = (-1)^{\#\text{ of flips}}$
- even/odd permutations
  ↳ $\operatorname{sign}(\sigma) = 1$
  → $\operatorname{sign}(\sigma) = -1$
- Notation $S_n = \{\text{all } \sigma \text{ permutations}\}$

Th
- # of permutations is $n!$
- every permutation is either even or odd.
- If $\tilde{\sigma}$ is a permutation similar to σ but has one more flip:

$$\operatorname{sign}(\tilde{\sigma}) = -\operatorname{sign}(\sigma)$$

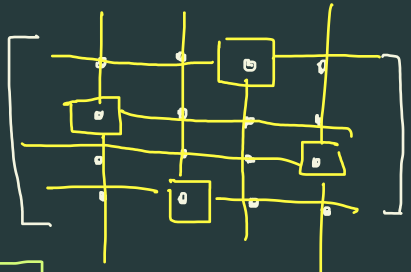
$\boxed{5.2}$ Summary

def

$$\boxed{\det(A) = \sum_{\sigma \in S_n} \overset{\text{"}\pm 1\text{"}}{\operatorname{sign}(\sigma)}\, \underbrace{a_{1\sigma(1)}\, a_{2\sigma(2)}\, a_{3\sigma(3)} \cdots\cdots a_{n\sigma(n)}}}$$

only one a_{ij} selection per row & per column!

where $A_{n\times n} = (a_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le n}}$



a single term in the sum is a product;

Properties

① A has entire row or column of zeros, then $\det(A) = 0$.

② $\det(A^T) = \det(A)$

③ A has a row that's proportional to another row (or column)

(or column) ex: [two rows are equal]

then $\boxed{\det(A) = 0}$

★useful★ ★④ A is upper- or lower-triangular matrix

(ie $a_{ij} = 0$ if $i > j$ upper) ($a_{ij} = 0$ if $i < j$ lower)

then $\boxed{\det(A) = a_{11}a_{22}\cdots\cdots a_{nn}}$ product of only diagonal entries

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

"upper triangular"

Determinants & EROs

Let E be an elementary matrix (ie one ERO to I_n)

① E = multiply i^{th} row by k : $\boxed{\det(E) = k}$ "think scale area/vol"

$\boxed{kR_i \to R_i}$

② E = exchanges row i & j : $\boxed{\det(E) = -1}$

$\boxed{R_i \leftrightarrow R_j}$

④ $\det(E) \neq 0$ for any elem. matrix.

③ Add k times row j to row i : $\boxed{\det(E) = 1}$

$\boxed{R_i + kR_j \to R_i}$

Ex $\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \boxed{4}$ $\quad 4R_2 \to R_2$

$\det\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \boxed{-1}$ $\quad R_1 \leftrightarrow R_3$

$\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} = \boxed{1}$ $\quad R_3 + (-3)R_1 \to R_3$

★ **Thm** If B is the result of a single ERO to A:

① $\boxed{\overset{A}{kR_i} \to \overset{B}{R_i}}$: $\boxed{\det(B) = k\det(A)}$

② $B : R_i \leftrightarrow R_j$: $\boxed{\det(B) = -\det(A)}$

③ $R_i + kR_j \to R_i$: $\boxed{\det(B) = \det(A)}$

$\underline{\text{Cor}}$ A similar result is true for $\underline{E}\ \underline{Col}\ \underline{O}$
(b/c $\det(A^T) = \det(A)$)

$\underline{\text{Ex}}$ Compute determinant of A using EROs & this theorem.

$$\det\begin{bmatrix} 8 & -2 & 3 & -7 \\ -3 & 0 & 4 & 8 \\ 6 & 2 & -1 & -5 \\ 5 & -9 & -2 & 9 \end{bmatrix} = \det\begin{bmatrix} 5 & -2 & 7 & 1 \\ 0 & 2 & 7 & 11 \\ 6 & 2 & -1 & -5 \\ 5 & -9 & -2 & 9 \end{bmatrix} = \det\begin{bmatrix} 5 & -2 & 7 & 1 \\ 0 & 2 & 7 & 11 \\ 6 & 2 & -1 & -5 \\ 0 & -7 & -9 & 8 \end{bmatrix}$$

- $R_1 + R_3 \to R_1$
- $2R_2 + R_3 \to R_2$
- $R_4 - R_1 \to R_4$
- $R_3 - R_1 \to R_3$

$$= \det\begin{bmatrix} 5 & -2 & 7 & 1 \\ 0 & 2 & 7 & 11 \\ 0 & 4 & -8 & -6 \\ 0 & -7 & -9 & 8 \end{bmatrix} = \frac{1}{2}\det\begin{bmatrix} 5 & -2 & 7 & 1 \\ 0 & 2 & 7 & 11 \\ 0 & 2 & -4 & -3 \\ 0 & -7 & -9 & 8 \end{bmatrix}$$

- $\frac{1}{2}R_3 \to R_3$
- $R_3 - R_2 \to R_3$

$$= \frac{1}{2}\det\begin{bmatrix} 5 & -2 & 7 & 1 \\ 0 & 2 & 7 & 11 \\ 0 & 0 & -11 & -14 \\ 0 & \boxed{-7} & -9 & 8 \end{bmatrix}$$

- $R_4 + 4R_2 \to R_4$

$$= \frac{1}{2}\det\begin{bmatrix} 5 & -2 & 7 & 1 \\ 0 & 2 & 7 & 11 \\ 0 & 0 & -11 & -14 \\ 0 & 1 & 19 & 52 \end{bmatrix} = \frac{2}{2}\det\begin{bmatrix} 5 & -2 & 7 & 1 \\ 0 & 2 & 7 & 11 \\ 0 & 0 & -11 & -14 \\ 0 & 0 & 31 & 93 \end{bmatrix}$$

- $2R_4 - R_2 \to R_4$
- $R_4 + \frac{31}{11}R_3 \to R_4$

$$= \det\begin{bmatrix} 5 & -2 & 7 & 1 \\ 0 & 2 & 7 & 11 \\ 0 & 0 & -11 & -14 \\ 0 & 0 & 0 & \frac{589}{11} \end{bmatrix}$$

So: $\det(A) = (5)(2)(-11)\left(\frac{589}{11}\right) = \boxed{-589}$

5.3 Summary

Thm • $A_{n\times n}$ is invertible $\boxed{\text{iff}}$ $\det(A) \neq 0$

• $A_{n\times n}$ is not invertible $\boxed{\text{iff}}$ $\det(A) = 0$

Sketch ($\Rightarrow$) A is invertible. $RREF(A) = I_n$. So

$\exists$ Elem. matrices: $E_1, E_2, \dots, E_k$

$$E_k \cdots E_3 E_2 E_1 A = I_n.$$

Since $\det(E_i) \neq 0$ & $\det(I_n) = 1$, we get:

$$\det(E_k \cdots E_1 A) = 1$$

$$\det(E_k) \cdot \det(E_{k-1} \cdots E_1 A) = 1$$

$$\vdots$$

$$\underbrace{\det(E_k) \cdots \det(E_1)}_{\text{non-zero!}} \cdot \det(A) = 1$$

so divide! so $\det(A) \neq 0$. $\square$

Thm • $\boxed{\det(A * B) = \det(A) * \det(B)}$

- $\det(A^k) = [\det(A)]^k$, $k > 0$ integer.

$A * A * \cdots * A = A^k$

- when A is invertible:

- $$\boxed{\det(A^{-1}) = \frac{1}{\det(A)} = \det(A)^{-1}}$$

- $\det(A^k) = [\det(A)]^k$, $k \in \mathbb{Z}$

Cofactor Expansion

- def i,j minor of A: determinant of the submatrix of A where delete i^{th} row & j^{th} colum,

↳ notation: $M_{ij}(A)$

i,j-cofactor of A: $C_{ij}(A) = (-1)^{i+j} M_{ij}(A)$

+	−	+	−
−	+	−	+
+	−	+	−
−	+	−	+

Thm Cofactor Formulas

- "along row i" $\boxed{\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}}$ (sum over j cols, i fixed)

- "along column j" $\boxed{\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij}}$ (sum over i rows, j fixed)

- $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$ (i fixed)
- $\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$ (j fixed)

Best of Both worlds

→ use EROs (when easy)
→ use cofactor expansion (when once you have enough zeros)

Ex Compute det(A) where $A = \begin{bmatrix} 8 & -2 & 3 & -7 \\ -3 & 0 & 6 & 8 \\ 6 & 2 & -1 & -5 \\ 5 & -9 & -2 & 9 \end{bmatrix}$ using any method.

- $R_1 + R_2 \to R_1$
- $R_4 - R_1 \to R_4$
- $R_3 + 2R_2 \to R_3$
- $R_2 + \frac{3}{5}R_1 \to R_2$

$$\det \begin{bmatrix} 5 & -2 & 9 & 1 \\ -3 & 0 & 6 & 8 \\ 0 & 2 & 11 & 11 \\ 0 & -7 & -11 & 8 \end{bmatrix} = \det \begin{array}{c} + \\ - \\ + \\ - \end{array} \begin{bmatrix} \boxed{5} & -2 & 9 & 1 \\ 0 & -6/5 & 57/5, & 43/5 \\ 0 & 2 & 11 & 11 \\ 0 & -7 & -11 & 8 \end{bmatrix}$$

$$6 + \frac{3}{5}(9) = \frac{57}{5}$$

$$8 + \frac{3}{5}(1) = \frac{43}{5}$$

now Cofactors

$$= 5\cdot(+1) \det \begin{bmatrix} -6/5 & 57/5 & 43/5 \\ 2 & 11 & 11 \\ -7 & -11 & 8 \end{bmatrix} = \det \begin{array}{ccc} + & - & + \\ \hline -6 & 57 & 43 \\ \hline 2 & 11 & 11 \\ -7 & -11 & 8 \end{array}$$

$$= -6 \det \begin{bmatrix} 11 & 11 \\ -11 & 8 \end{bmatrix} - 57 \det \begin{bmatrix} 2 & 11 \\ -7 & 8 \end{bmatrix} + 43 \det \begin{bmatrix} 2 & 11 \\ -7 & -11 \end{bmatrix}$$

$$= -6 (88 + 121) - 57 (16 + 77) + 43 (-22 + 77)$$

$$= \boxed{-4190}$$