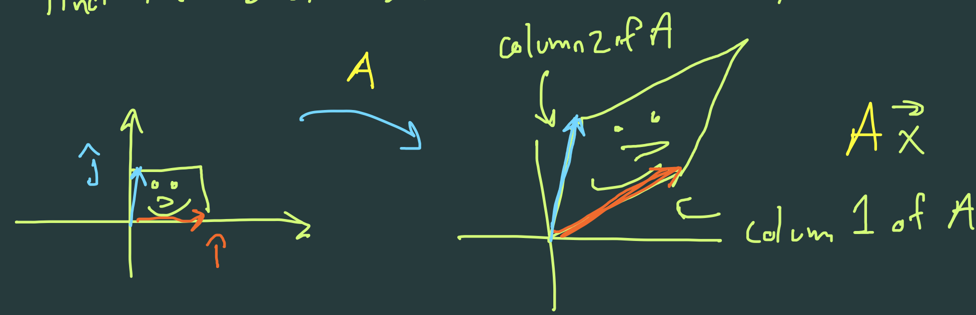
Chapter 6
Eigentheory

6.1
Eigentheory
for Matrices

Motivation Given a matrix $A_{n\times n}$, square, or $T: \mathbb{R}^n \to \mathbb{R}^n$ (linear operator)

Goal find the "best" basis matched to A/T



geometrically look for vectors that get only stretched by A
but keep same direction

6.1

def • λ is an eigenvalue of A (equiv. $T: \mathbb{R}^n \to \mathbb{R}^n$) if there's a solution to

$$\boxed{A\vec{x} = \lambda\vec{x}}$$

• $\vec{v}_\lambda$ is an eigenvector of A ($\sim T$) if $\vec{v}_\lambda \neq \vec{0}_n$ (in $\mathbb{R}^n$) is a solution to

$$\boxed{A\vec{v}_\lambda = \lambda\vec{v}_\lambda}$$

How to find λ & $\vec{v}_\lambda$?

- Start by assuming we have a λ:

$$A\vec{x} = \lambda\vec{x}$$

iff $\quad A\vec{x} = \lambda(I_n\vec{x})$

iff $\quad A\vec{x} = \lambda I_n\vec{x}$

iff $\quad A\vec{x} - \lambda I_n\vec{x} = \vec{0}_n$

Eigenvector

HSOE: $\quad \underbrace{(A-\lambda I_n)\vec{x} = \vec{0}_n}_{\text{has non-trivial solution}} \longleftrightarrow \underbrace{\vec{x} \in NS(A-\lambda I_n)}_{\text{has dimension} \geq 1}$

$\boxed{\text{iff}}$ $\qquad$ iff

$A-\lambda I_n$ is not invertible $\qquad$ RREF has row 0's

Using determinants, also get:

$\boxed{\text{iff}}$

$$\boxed{\det(A-\lambda I_n)=0}$$

called characteristic Equation

degree n polynomial in λ

so want to find roots of charact. poly.

finding eigenvalues λ

Characteristic polynomial: $\boxed{p_A(\lambda) = \det(A-\lambda I_n)}$

Case 2x2: $p_A(\lambda)$ = degree 2 polynomial $\in \mathbb{P}^2$

Δ = discriminant $= b^2-4ac$

Δ Test

- $\Delta = 0$: one real sol
- $\Delta < 0$: two $\mathbb{C}$ solutions
- $\Delta > 0$: 2 real sol
  - $\mathbb{Z}$
  - $\mathbb{Q}$
  - irrational

Ex $A = \begin{bmatrix} 2 & 2 \\ -4 & 8 \end{bmatrix}$ a) find eigenvalues
b) find corresp. eigenvectors.

- $p_A(\lambda) = \det(A - \lambda I_2) = \det\left(\begin{bmatrix} 2 & 2 \\ -4 & 8 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$

$$= \det\left(\begin{bmatrix} 2-\lambda & 2 \\ -4 & 8-\lambda \end{bmatrix} \right) = (2-\lambda)(8-\lambda) + 8.$$

- λ eigenvalues are roots $p_A(\lambda) = 0$:

$$16 - 2\lambda - 8\lambda + \lambda^2 + 8 = 0 \longleftrightarrow \lambda^2 - 10\lambda + 24 = 0$$

$$(\lambda - 4)(\lambda - 6) = 0$$

$$\boxed{\lambda = 4, 6}$$

- $\lambda = 4$: $A\vec{x} = 4\vec{x}$ iff $NS\left(\begin{bmatrix} 2-4 & 2 \\ -4 & 8-4 \end{bmatrix} \right) = NS\left(\begin{bmatrix} -2 & 2 \\ -4 & 4 \end{bmatrix} \right)$.

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ -4 & 4 & 0 \end{array}\right] \xrightarrow{R_2 - 2R_1 \to R_2} \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2}R_1 \to R_1} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

x = leading
y = free

$\begin{cases} x - y = 0 \\ 0 = 0 \end{cases}$ $\quad x = y$, $y = y$ $\quad \vec{x} = \begin{bmatrix} y \\ y \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $y \in \mathbb{R}$.

so $\boxed{\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$ is an eigenvector corresponding to $\lambda = 4$.

Remark other eigenvectors are: $\begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} \pi \\ \pi \end{bmatrix}, \ldots$ etc

- $\lambda = 6$: $\begin{bmatrix} 2-6 & 2 \\ -4 & 8-6 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ -4 & 2 \end{bmatrix} \xrightarrow[R_2 - R_1 \to R_1]{-\frac{1}{4}R_1 \to R_1} \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$

$$\begin{cases} x - \frac{1}{2}y = 0 \\ y = y \end{cases} \qquad \begin{matrix} x = \frac{1}{2}y \\ y = y \end{matrix} \qquad \vec{x} = \begin{bmatrix} \frac{1}{2}y \\ y \end{bmatrix} = y \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

So $\vec{v} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 6$.

Better: $\boxed{\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$

Ex $A = \begin{bmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{bmatrix}$ check if $\vec{v} = \begin{bmatrix} 16 \\ 4 \\ 1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

these are eigenvectors.

- $A\vec{v} = \lambda\vec{v}$?

yes!

& $\lambda = 2$ is an eigenvalue of A!

$$A\vec{v} = \begin{bmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 16 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 32 \\ 8 \\ 2 \end{bmatrix}$$

$$\lambda\vec{v} = \begin{bmatrix} 16\lambda \\ 4\lambda \\ \lambda \end{bmatrix} = \begin{bmatrix} 32 \\ 8 \\ 2 \end{bmatrix} \implies \checkmark \checkmark \ \underline{\lambda = 2} \checkmark$$

- $A\vec{w} = \lambda\vec{w}$?

NO!

$$A\vec{w} = \begin{bmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 28 \\ 1 \\ 1 \end{bmatrix} \overset{?}{=} \lambda \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

impossible! $\lambda = 1/2$

$28 = \lambda 2$ no good!

Ex $\lambda = 0$ can be an eigenvalue!

$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$. $A\vec{v} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$= 0 \cdot \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

<u>Sometimes</u>: $\vec{v} \in \ker(A)$ can be an eigenvector only when $\lambda = 0$ is an eigenvalue.

Def Eigen Spaces

$$Eig(A,\lambda) = \{ \vec{v} \in \mathbb{R}^n \mid A\vec{v} = \lambda \vec{v} \}$$

$\boxed{\text{Thm}}$ a) $Eig(A,\lambda) = NS(A - \lambda I_n)$. ✓ (Useful for computations)

b) $Eig(A,\lambda) \trianglelefteq \mathbb{R}^n$ (a subspace).

c) when λ is an eigenvalue of A, Eigenspace is not $\{\vec{0}\}$ — the trivial space. ✓

d) If $Eig(A,\lambda)$ is $\{\vec{0}\}$ then λ is not a eigenvalue of A. ✓

e) If λ is an eigenvalue of A, then $\dim(Eig(A,\lambda)) \geq 1$. ✓

Pf (b). NTS 1) non-empty

2) CVA

3) CSM

1) $\vec{0} \in Eig(A,\lambda)$. True b/c $A\vec{0} = \vec{0}$ & $\lambda\vec{0} = \vec{0}$ so $A\vec{0} = \lambda\vec{0}$ ✓

2) Let $\vec{v}_1, \vec{v}_2 \in Eig(A,\lambda)$. Then

$$A\vec{v}_1 = \lambda \vec{v}_1 \quad \& \quad A\vec{v}_2 = \lambda \vec{v}_2$$

So: $A(\vec{v}_1 + \vec{v}_2) = A\vec{v}_1 + A\vec{v}_2 = \lambda\vec{v}_1 + \lambda\vec{v}_2 = \lambda(\vec{v}_1 + \vec{v}_2)$.

Then $\vec{v}_1 + \vec{v}_2 \in Eig(A,\lambda)$.

3) let $\vec{v} \in Eig(A,\lambda)$, let $r \in \mathbb{R}$:

$$A\vec{v} = \lambda\vec{v}$$

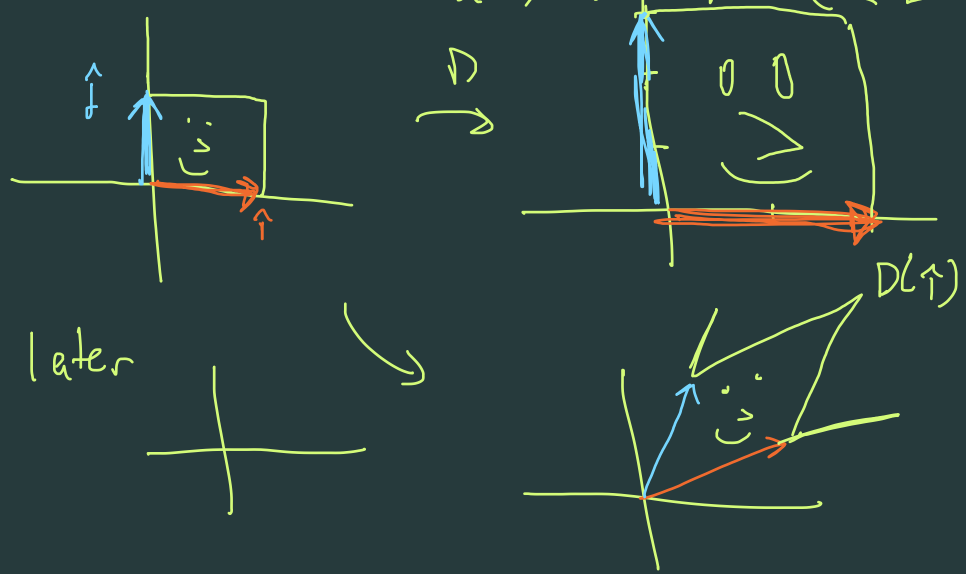
so $$A(r\vec{v}) = rA\vec{v} = r(\lambda\vec{v}) = (r\lambda)\vec{v} = \lambda(r\vec{v}).$$

Thus, $r\vec{v} \in Eig(A,\lambda)$. $\square$

Ex $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ $\quad \lambda = 2, 3$

$Eig(D, 2) = Span\left(\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}\right)$

$Eig(D, 3) = Span\left(\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}\right).$



Thm A is upper-or-lower-triangular. Then

a) $P_A(\lambda) = (c_1 - \lambda)(c_2 - \lambda) \cdots (c_n - \lambda)$

where c_i are the entries in the main diagonal.

Key Also

b) when D is diagonal & $D = diag(c_1, c_2, \dots, c_n)$

Then $c_1, \dots, c_n$ are eigenvalues of D

Pf $$P_A(\lambda) = \det\left[\begin{pmatrix} c_1 & c_2 & & * \\ & & \ddots & \\ 0 & & & c_n \end{pmatrix} - \lambda \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\right]$$

Assume A is upper.

$$= \det \begin{bmatrix} c_1 - \lambda & & * \\ & c_2 - \lambda & \\ & & \ddots & \\ 0 & & & c_n - \lambda \end{bmatrix} \underset{\text{Thm Ch5}}{=} (c_1 - \lambda) \cdots (c_n - \lambda).$$

$\square$

Ex $A = \begin{bmatrix} 5 & 13 & -6 \\ 0 & -2 & 3 \\ 0 & 0 & 5 \end{bmatrix}$, find eigenvalues & corresp. eigenspaces.

Sol $\bullet$ $p(\lambda) = (5-\lambda)(\lambda+2)(5-\lambda) = (5-\lambda)^2(\lambda+2)$.

$\lambda = 5, -2$

$\bullet$ $\underline{\lambda = -2}$ $\quad NS\left(\begin{bmatrix} 5+2 & 13 & -6 \\ 0 & -2+2 & 3 \\ 0 & 0 & 5+2 \end{bmatrix}\right)$

$$\begin{bmatrix} 7 & 13 & -6 \\ 0 & 0 & 3 \\ 0 & 0 & 7 \end{bmatrix} \xrightarrow{R_3 - 2R_2 \to R_3} \begin{bmatrix} 7 & 13 & -6 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 - 3R_3 \to R_2} \begin{bmatrix} 7 & 13 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 7 & 13 & -6 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$7x_1 + 13x_2 - \cancel{6x_3} = 0$

$x_3 = 0$

$x_2 = 0$

$x_1 = -13/7\, x_2$

$x_2 = x_2$

$x_3 = 0$

$\vec{x} = \begin{bmatrix} -13/7\, x_2 \\ x_2 \\ 0 \end{bmatrix}$

leading: x_1, x_3

free: x_2

so $\vec{x} = x_2 \begin{bmatrix} -13/7 \\ 1 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -13 \\ 7 \\ 0 \end{bmatrix}, \quad x_2 \in \mathbb{R}$

$Eig(A, -2) = Span\left(\left\{\begin{bmatrix} -13 \\ 7 \\ 0 \end{bmatrix}\right\}\right)$.

$\bullet$ $\underline{\lambda = 5}$ $\quad NS\left(\begin{bmatrix} 0 & 13 & -6 \\ 0 & -7 & 3 \\ 0 & 0 & 0 \end{bmatrix}\right)$

$$\begin{bmatrix} 0 & 13 & -6 \\ 0 & -7 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 + 2R_2 \to R_1]{} \begin{bmatrix} 0 & -1 & 0 \\ 0 & -7 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_2 - 7R_1 \to R_2]{} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[-R_1 \to R_1]{} \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{cases} x_2 = 0 \\ x_3 = 0 \\ x_1 = \text{free} \end{cases}$$

$\vec{x} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$

$Eig(A, 5) = Span\left(\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}\right)$.

Ex $A = \begin{bmatrix} 5 & 14 & -6 \\ 0 & -2 & 3 \\ 0 & 0 & 5 \end{bmatrix}$ but 13 changed to a 14!

Sol $\lambda = 5, -2$ but eigenspaces could be different!

$\lambda = 5$

$$\begin{bmatrix} 0 & 14 & -6 \\ 0 & -7 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 + 2R_2 \to R_1]{} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -7 & 3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ 3/7\, x_3 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 3/7 \\ 1 \end{bmatrix}$$

leading: x_2

free: x_1, x_3

$-7x_2 + 3x_3 = 0$

$x_2 = \frac{-3}{-7} x_3$

$$\text{Eigen}(A, 5) = \text{Span}\left(\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 7 \end{bmatrix} \right\} \right).$$

$\lambda = -2$ $\text{Eig}(A, -2) = \text{Span}\left(\left\{ \begin{bmatrix} -14 \\ 1 \\ 0 \end{bmatrix} \right\} \right).$