

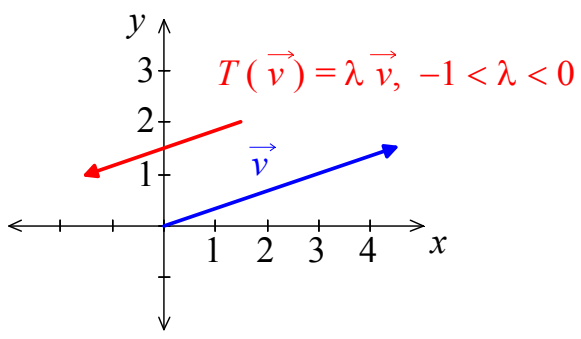
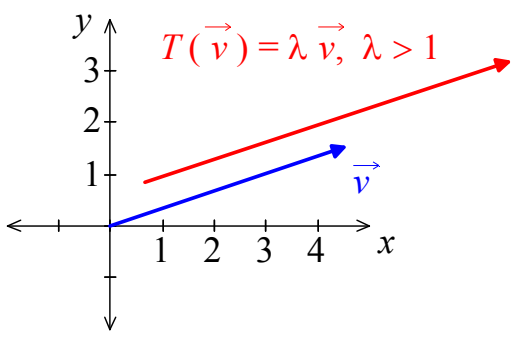
6.1 Eigentheory of Matrices

Definition: Let A be an $n \times n$ matrix. We say that $\lambda \in \mathbb{R}$ (the Greek letter *lambda*) is an **eigenvalue** of A , and a **non-zero** vector $\vec{v} \in \mathbb{R}^n$ is an **eigenvector** for A associated to λ , or simply an eigenvector for λ , if:

$$A\vec{v} = \lambda\vec{v}.$$

$A\vec{v}$ = parallel to \vec{v} but scales it by λ

In other words, if $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$, with $[T] = A$, then $T(\vec{v})$ is **parallel** to \vec{v} :



How do we Find Eigenvalues and Eigenvectors?

$$A\vec{v} = \lambda\vec{v} = (\lambda I_n)(\vec{v})$$

$$(\lambda I_n)(\vec{v}) - A\vec{v} = \vec{0}_n$$

$$(\lambda I_n - A)\vec{v} = \vec{0}_n$$

Definition/Theorem: Let A be an $n \times n$ matrix. Then we can find a real number λ and a non-zero vector $\vec{v} \in \mathbb{R}^n$ such that:

$$A\vec{v} = \lambda\vec{v}$$

if and only if $\det(\lambda I_n - A) = 0$.

The equation above is called the **characteristic equation** of the matrix A .

The determinant in this equation is a polynomial whose highest term is λ^n , and it is called the **characteristic polynomial** of A , denoted $p_A(\lambda)$, or $p(\lambda)$:

$$p_A(\lambda) = p(\lambda) = \det(\lambda I_n - A).$$

The *highest term* comes from:

$$\operatorname{sgn}(\sigma)(\lambda - a_{1,1}) \cdot (\lambda - a_{2,2}) \cdot \cdots \cdot (\lambda - a_{n,n})$$

But for this term, $\sigma = (1, 2, \dots, n)$

$$\operatorname{sgn}(\sigma) = 1.$$

Possibilities for 2×2 Matrices

Suppose A has integer or rational entries only.

The quadratic characteristic polynomial will have integer or rational coefficients.

It can have:

- *two distinct roots*, (if the discriminant is not 0) or
- a *single* (repeated or double) root (if the discriminant is 0).

If the roots are *distinct*, then they could be:

- *imaginary*, (if the discriminant is negative)
- *irrational*, (if the discriminant is a positive non-square)
- *integer* or *rational*, (if the discriminant is a positive perfect square)

Example:

$$\begin{bmatrix} 285 & 504 \\ -160 & -283 \end{bmatrix}$$

Eigenspaces

How to compute,

Definition/Theorem — Eigenspaces:

Let A be an $n \times n$ matrix and $\lambda \in \mathbb{R}$. We define the **eigenspace** of A associated to λ , denoted $Eig(A, \lambda)$, to be:

$$Eig(A, \lambda) = \{ \vec{v} \in \mathbb{R}^n \mid A\vec{v} = \lambda\vec{v} \} = NS(A - \lambda I_n)$$

when λ given:
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Notice that $A\vec{0}_n = \vec{0}_n = \lambda\vec{0}_n$, so $\vec{0}_n \in Eig(A, \lambda)$.

If λ is an actual **eigenvalue** for A , then $Eig(A, \lambda) = nullspace(A - \lambda I_n)$, which is a non-zero subspace of \mathbb{R}^n containing all the **eigenvectors** of A associated to λ , and thus its dimension is strictly positive.

If λ is **not** an eigenvalue of A , then $Eig(A, \lambda)$ consists only of $\vec{0}_n$. In this case, we can refer to $Eig(A, \lambda)$ as a **trivial eigenspace**.

Thus, we can say that λ is an **eigenvalue** of A *if and only if* the eigenspace $Eig(A, \lambda)$ is at least **1-dimensional**.

Eigentheory for Triangular Matrices

Theorem: Let A be an upper (or) lower **triangular** $n \times n$ matrix, and suppose the entries along the main diagonal are c_1, c_2, \dots, c_n . Then: the **characteristic polynomial** of A is:

$$p(\lambda) = (\lambda - c_1)(\lambda - c_2) \cdots (\lambda - c_n),$$

and therefore the **eigenvalues** are precisely c_1, c_2, \dots, c_n .

Moreover, if:

$$D = \text{Diag}(d_1, d_2, \dots, d_n)$$

is a **diagonal** matrix, then for every $i = 1 \dots n$: e_i is an **eigenvector** for d_i .

Ex $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ eigenvalues are $\lambda = 3, 5, -2$.

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- $\text{Eig}(D, 3) = \text{Span}(\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\})$
- $\text{Eig}(D, 5) = \text{Span}(\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\})$
- $\text{Eig}(D, -2) = \text{Span}(\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\})$

Example:

$$\begin{bmatrix} 5 & 13 & -6 \\ 0 & -2 & 3 \\ 0 & 0 & 5 \end{bmatrix}$$

Follow up: what happens if the 14 is turned into 13?