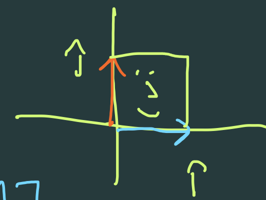
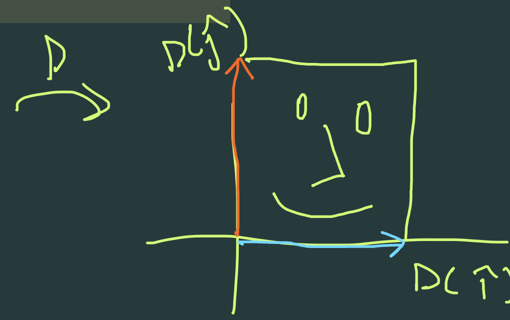
Ex $\boxed{D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}}$.





$D(\hat{i}) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\hat{i}$ ✓

$D(\hat{j}) = \begin{bmatrix} 0 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3\hat{j}$ ✓

$\boxed{\text{Diagonal matrices are awesome!}}$

→ simple geometrically

→ read off eigenvalues:

$\lambda = 2, 3$

→ read off eigenspaces:

$Eig(D, 2) = Span(\hat{i})$

$Eig(D, 3) = Span(\hat{j})$

→ $D^2 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$

$= \begin{bmatrix} 2^2 & 0 \\ 0 & 3^2 \end{bmatrix}$

$D^k = \begin{bmatrix} 2^k & 0 \\ 0 & 3^k \end{bmatrix}$

Since in $\mathbb{R}^2$

- $Eig(D,2) \oplus Eig(D,3) = \mathbb{R}^2$
- $Eig(D,3)^{\perp} = Eig(D,2)$
- $Eig(D,2)^{\perp} = Eig(D,3)$

Ex $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\lambda = 1, 2, 3$

$Eig(D, 1) = Span\left(\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}\right)$.

$D^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1^2 & 0 & 0 \\ 0 & 2^2 & 0 \\ 0 & 0 & 3^2 \end{bmatrix}$

Diagonalizable Matrices

def $A \in \mathbb{M}_{n\times n}$ is diagonalizable if $\exists$ invertible matrix $C \in \mathbb{M}_{n\times n}$

so that

$$\boxed{C^{-1} * A * C = D} = \text{a diagonal matrix!}$$

where D is a diagonal matrix $D = \text{diag}(\alpha_1, \alpha_2, \dots, \alpha_n)$

- $C^{-1}AC$ is called the conjugate of A by C
- When A is not diagonalizable we call it defective.
- We say A & D are similar matrices (the conjugates of each other).
  ↳ this is more general notion than diagonalizable.

A diagonalizable $\boxed{\text{iff}}$ $C^{-1}AC = D$ $\boxed{\text{iff}}$ $AC = CD$

$$C(C^{-1}AC) = CD$$
$$\underbrace{(CC^{-1})}_{I}AC = CD$$
$$AC = CD$$

$$= \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_n \end{bmatrix}$$

$$= \begin{bmatrix} c_{11}\alpha_1 & & c_{1n}\alpha_n \\ c_{21}\alpha_1 & \cdots & c_{2n}\alpha_n \\ \vdots & & \vdots \\ c_{n1}\alpha_1 & & c_{nn}\alpha_n \end{bmatrix}$$

$$\left[A\vec{c}_1 \,\middle|\, \cdots \,\middle|\, A\vec{c}_n \right] = \left[\vec{c}_1\alpha_1 \,\middle|\, \cdots \,\middle|\, \vec{c}_n\alpha_n \right]$$

where $\vec{c}_i$ is the i^{th} column of C

$\boxed{\text{iff}}$ $\boxed{A\vec{c}_i = \alpha_i \vec{c}_i}$

This says: α_i's are eigenvalues of A

$\vec{c}_i$'s are eigenvectors of A correspond to α_i

Moreover: C is invertible!

↳ columns $\vec{c}_i$ are LI linearly independent!

Ex $A = \begin{bmatrix} -12 & 15 \\ -10 & 13 \end{bmatrix}$. $C = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$, $C^{-1} = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$

Is $C^{-1}AC$ the conjugate of A by C?

Sol If yes, we should get a diagonal matrix:

$$C^{-1}AC = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -12 & 15 \\ -10 & 13 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & -6 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \leftarrow \text{diagonal!}$$

conclusions: 1) $\lambda = 3, -2$ are eigenvalues of A.

2) columns of C are eigenvectors corresponding to e.vals.

- $Eig(A, 3) = Span(\text{first col } C) = Span\left(\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}\right)$
- $Eig(A, -2) = Span(2^{nd} \text{ col } C) = Span\left(\left\{\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right\}\right)$.

Check $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 & 15 \\ -10 & 13 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Rmk Can get A from D, C, C^{-1}:

$$\boxed{A = CDC^{-1}}$$

Thm $A_{n \times n}$ has imaginary eigenvalues (ie in $\mathbb{C}$).
Then A is not diagonalizable over $\mathbb{R}$.

Idea if it is diag, then it has n real eigenvalues in $\mathbb{R}$ on the diagonal

form C = columns are eigenvectors corresponding to eigenvalues. Check C^{-1} exists and $C^{-1}AC = D$. So A can't have any $\mathbb{C}$ eigenvalues.

Thm Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ ordered eigenvectors corresponding to eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$.

- If the λ_i's are distinct, then S is linearly independent.
- Thus, if A has m distinct eigenvalues, $\exists$ at least m linearly independent eigenvectors for A.

Rmk General result on Eigentheory.

Pf Use induction on k.

Case $k=1$.

$S = \{\vec{v}_1\}$. Since $\vec{v}_1$ is an eigenvector, $\vec{v}_1 \neq \vec{0}$. Since S one vectors set & non-zero, it is LI.

IH : Assume that $S = \{\vec{v}_1, \ldots, \vec{v}_{k-1}\}$ is linearly independent.

NTS $S \cup \{\vec{v}_k\} = S'$ is also LI.

By contradiction.

Suppose that $S' = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is LD.

Then $\exists$ $c_1, \ldots, c_k \in \mathbb{R}$ not all of them zero so that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}. \quad (*)$$

Then

$$A(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) = A\vec{0}$$

$$c_1 A\vec{v}_1 + c_2 A\vec{v}_2 + \cdots + c_k A\vec{v}_k = \vec{0} \qquad \text{(Linearity properties of MM)}$$

$$c_1(\lambda_1\vec{v}_1) + c_2(\lambda_2\vec{v}_2) + \cdots + c_k(\lambda_k\vec{v}_k) = \vec{0} \quad (**)$$

- Multiply $(*)$ by $\boxed{\lambda_k}$:

$(c_1\lambda_k)\vec{v}_1 + \cdots + c_k\lambda_k\vec{v}_k = \vec{0}$] (***)

Subtracting (***) & (**):

$$\boxed{c_1(\lambda_1-\lambda_k)\vec{v}_1 + c_2(\lambda_2-\lambda_k)\vec{v}_2 + \cdots + c_{k-1}(\lambda_{k-1}-\lambda_k)\vec{v}_{k-1} = \vec{0}}$$

$k-1$

Case $c_k = 0$:

Then $c_1\vec{v}_1 + \cdots + c_{k-1}\vec{v}_{k-1} = 0$ & since c_i's not all zero

This contradicts that $\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$ is LI. This can't happen.

Case $c_k \neq 0$.

(*) says $\vec{v}_k \in Span(\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$

$\vec{v}_k = d_1\vec{v}_1 + \cdots + d_{k-1}\vec{v}_{k-1}$

Replace this in.

Since $\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$ are independent $\Rightarrow$ $c_i(\lambda_i - \lambda_k) = 0$, $i = 1, \dots, k-1$

& λ_i's are distinct, so $\lambda_i - \lambda_k \neq 0$, $i = 1, 2, \dots, k-1$.

Then $c_i = 0$ $\forall i = 1, \dots, k-1$.

This contradicts that c_i's are not all zero! $\square$

Ex $A = \begin{bmatrix} 2 & -9 & 6 \\ 0 & 5 & -2 \\ 0 & 0 & 2 \end{bmatrix}$ Recall $p(\lambda) = (2-\lambda)^2(5-\lambda)$

↳ $\lambda = 2, 5$ eigenvalues.

only 2 distinct ei.val. Is A diag?

Need to find $Eig(A, 2)$ & $Eig(A, 5)$

• $\lambda = 2$ $NS\left(\begin{bmatrix} \boxed{2-2} & -9 & 6 \\ 0 & \boxed{5-2} & -2 \\ 0 & 0 & \boxed{2-2} \end{bmatrix} \right)$

$$\begin{bmatrix} 0 & -9 & 6 \\ 0 & 3 & -2 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} x_1 = x_1 \\ 3x_2 = 2x_3 \\ x_3 = x_3 \end{array} \quad \vec{x} = \begin{bmatrix} x_1 \\ 2/3\, x_3 \\ x_3 \end{bmatrix}$$

$$\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 2/3 \\ 1 \end{bmatrix}$$

$$Eig(A, \lambda) = Span\left(\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right\} \right)$$

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$ LI? yes!

• $\lambda = 5$

$$\begin{bmatrix} -3 & -9 & 6 \\ 0 & 0 & -2 \\ 0 & 0 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} -3 & -9 & 6 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} -3 & -9 & 6 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} -3 & -9 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} -3x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

x_2 free

$$Eig(A, 5) = Span\left(\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right\} \right).$$

so we get 3 LI eigenvectors $\longrightarrow$ A is diag!

$\boxed{\text{Thm}}$ A is diag $\boxed{\text{iff}}$ geometric mult. = algebraic mult.

Thm A has n real distinct! eigenvalues. Then A is diagonalizable.

Pf Let $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ be distinct! eigenvalues. For each i, $\vec{v}_i \in Eig(A, \lambda_i)$

be a corresponding eigenvectors.

Since λ_i's are distinct:

Lemma If $\lambda_i \neq \lambda_j$ eigenvalues of A then

$$\boxed{Eig(A, \lambda_i) \cap Eig(A, \lambda_j) = \{\vec{0}\}}$$

Pf If $\vec{v}$ is in the intersection, then $\vec{v} \in Eig(A, \lambda_i)$ & $\vec{v} \in Eig(A, \lambda_j)$.

$$A\vec{v} = \lambda_i \vec{v} \quad \& \quad A\vec{v} = \lambda_j \vec{v}$$

Then $\lambda_i \vec{v} = \lambda_j \vec{v} \implies (\lambda_i - \lambda_j)\vec{v} = \vec{0}$

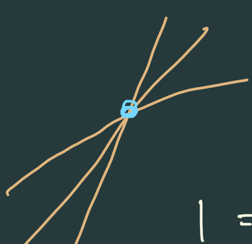
ZFT $\implies \lambda_i - \lambda_j = 0$ or $\vec{v} = \vec{0}$.

But $\vec{v} \neq \vec{0}$ b/c eigenvector so $\lambda_i = \lambda_j$. which contradicts $\lambda_i \neq \lambda_j$. $\square$

By Lemma, $\forall i \neq j$: $Eig(A, \lambda_i) \cap Eig(A, \lambda_j) = \{\vec{0}\}$.

Also: $\dim(Eig(A, \lambda_i)) \geq 1$, since λ_i is an eigenvalue.

So since $Eig(A, \lambda_i) \leq \mathbb{R}^n$ & we have n of them we must have:



$$\dim(Eig(A, \lambda_i)) = 1 \quad \text{for each } i = 1, \dots, n.$$

$$1 = \dim(Eig(A, \lambda_i)) \underbrace{\overset{\text{"Deep Thm"}}{\leq} 1}_{\text{b/c } \lambda_i\text{'s are distinct}} = \text{alg. mult of } \lambda_i$$

$$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

so algebraic mult is 1 for each!

Thus, geo. mult $= 1 =$ alg. mult! So A is diagonal by previous thm. $\square$

<u>Ex</u> $A = \begin{bmatrix} -1 & -2 & 2 \\ 2 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ w/ $p(\lambda) = \lambda(\lambda-1)(\lambda-3)$

$\hookrightarrow$ $\lambda = 0, 1, 3$ eigenvalues. (distinct!) real!

- $\lambda = 0$ is eigen-val. $\Rightarrow \ker(A) = Eig(A, 0) \neq \{\vec{0}\} \Rightarrow A$ <u>not</u> invertible.
- But it is diagonalizable!
- To find C = put eigenvectors as columns!
    - $\lambda = 0$: $Eig(A, 0) = Span\left(\left\{\begin{bmatrix} -6 \\ 5 \\ 2 \end{bmatrix}\right\}\right)$
    - $\lambda = 1$: $Eig(A, 1) = Span\left(\left\{\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}\right\}\right)$
    - $\lambda = 3$: $Eig(A, 3) = Span\left(\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}\right)$.

By lemma, these vectors are LI:

$$C = \begin{bmatrix} -6 & -2 & 0 \\ 5 & 3 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

$$C^{-1} = \begin{bmatrix} -1/3 & -1/3 & 1/3 \\ 1/2 & 1 & -1 \\ 1/6 & -1/3 & 4/3 \end{bmatrix}.$$

Should get:

$$C^{-1}AC = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$\boxed{\text{Thm}}$ If A is diagonalizable, then $A^k = CD^kC^{-1}$

$$= C \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix} C^{-1}$$