6.3 Diagonalization of Square Matrices

Definition: Let A be an $n \times n$ matrix. We say that A is ***diagonalizable*** if we can find an ***invertible*** matrix C such that:

$$C^{-1}AC = D,$$

where $D = Diag(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a diagonal matrix, or equivalently:

$$AC = CD \quad \text{or} \quad A = CDC^{-1}$$

We also say that C ***diagonalizes*** A.

When Can We Diagonalize?

Study:

$$AC = CD$$

Partition C into ***columns***:

$$C = \left[\ \vec{v}_1 \mid \vec{v}_2 \mid \cdots \mid \vec{v}_n\ \right]$$

$$AC = \left[\ A\vec{v}_1 \mid A\vec{v}_2 \mid \cdots \mid A\vec{v}_n\ \right]$$

$$CD = \left[\alpha_1\vec{v}_1 \mid \alpha_2\vec{v}_2 \mid \cdots \mid \alpha_n\vec{v}_n\right]$$

We must satisfy:

$$A\vec{v}_i = \alpha_i\vec{v}_i$$

for each column $\vec{v}_i$.

The Basis Test for Diagonalizability

Theorem (The Basis Test for Diagonalizability):

Let A be an $n \times n$ matrix. Then, A is diagonalizable ***if and only if*** we can find a ***basis*** for $\mathbb{R}^n$ consisting of n ***linearly independent eigenvectors*** for A, say $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. If this is the case, then the diagonalizing matrix C is simply the matrix whose ***columns*** are $\vec{v}_1$, $\vec{v}_2$, $\ldots$, $\vec{v}_n$, and the diagonal matrix D contains the corresponding ***eigenvalues*** along the main ***diagonal***.

Keep It Real

Theorem: Let A be an $n \times n$ matrix with imaginary eigenvalues. Then A is not diagonalizable over the set of ***real*** invertible matrices.

Independence of Eigenvectors

Theorem: Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be an ordered set of eigenvectors for an $n \times n$ matrix A, and suppose that the corresponding eigenvalues λ_1, λ_2, $\ldots$, λ_k for these eigenvectors are all ***distinct***. Then: S is ***linearly independent***. Thus, if A has a total of m distinct eigenvalues, we can find ***at least*** m linearly independent eigenvectors for A.

Use induction on k.

$k = 1$: Why is $\{\vec{v}_1\}$ independent?

Inductive Hypothesis: Assume $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_j\}$ is independent.

Inductive Step: Prove $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_j, \vec{v}_{j+1}\}$ is still independent:

Geometric and Algebraic Multiplicities

Definitions: Let A be an $n \times n$ matrix with ***distinct*** (possibly imaginary) eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Suppose $p(\lambda)$ factors as:

$$p(\lambda) = (\lambda - \lambda_1)^{n_1} \cdot (\lambda - \lambda_2)^{n_2} \cdot \cdots \cdot (\lambda - \lambda_k)^{n_k},$$

where $\boxed{n_1 + n_2 + \cdots + n_k = n.}$

We call the exponent n_i the ***algebraic multiplicity*** of λ_i.

We call $\boxed{dim(Eig(A, \lambda_i))}$ the ***geometric multiplicity*** of λ_i.

We agree that $dim(Eig(A, \lambda_i)) = 0$ if λ_i is an ***imaginary*** eigenvalue.

A Deep Theorem from "Algebraic Geometry"

Theorem (The Geometric vs. Algebraic Multiplicity Theorem):

For any eigenvalue λ_i of an $n \times n$ matrix. A, the ***geometric multiplicity*** of λ_i is ***at most equal*** to the ***algebraic multiplicity*** of λ_i.

$$\boxed{\text{geometric mult} \leq \text{algebraic mult}}$$

$$1 \leq \dim(Eig(A, \lambda_i)) \leq n_i$$

if λ_i's an eigenvalue.

Consequently:

Theorem (The Multiplicity Test for Diagonalizability):

Let A be an $n \times n$ matrix. Then A is diagonalizable ***if and only if*** for all of its eigenvalues λ_i, the geometric multiplicity of λ_i is ***exactly equal*** to its algebraic multiplicity.

A Sure Bet

Theorem: Let A be an $n \times n$ matrix with n ***distinct (real) eigenvalues***. Then A is diagonalizable.

Powers of Diagonalizable Matrices

$$A = CDC^{-1}$$

$$\begin{aligned} A^2 &= (CDC^{-1})(CDC^{-1}) \\ &= CD(C^{-1}C)DC^{-1} \\ &= CD^2C^{-1} \end{aligned}$$

$$A^k = CD^kC^{-1}$$