7.1 Inner Product Spaces

Def $V = C([a,b], \mathbb{R}) = \{ f : [a,b] \to \mathbb{R} \mid \text{continuous} \}$.

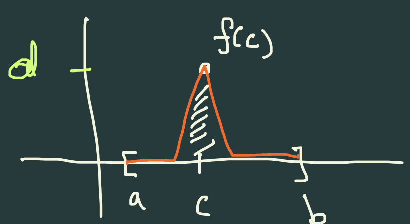
$$\langle f, g \rangle = \int_a^b f(x) g(x)\, dx.$$

Thm This is an inner product on V.

Pf
- Symmetry: ✓
- Add: ✓
- Homog.: ✓
- Positivity: Assume $f \in V$ and $f \neq$ zero function.

  $f : [a,b] \to \mathbb{R}$ & continuous.

  NTS $\langle f, f \rangle > 0$. iff $\boxed{\int_a^b f^2(x)\, dx > 0}$

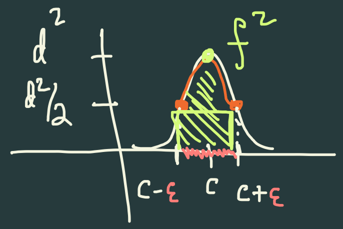


Since f is not the zero function, $\exists\, c \in (a,b)$ where $f(c) \neq 0$. WLOG: $f(c) > 0$. Call it d, ie $f(c) = d$. So $d > 0$.

Then $f^2(c) = d^2 > 0$.

Now, f^2 is continuous. There's exists $\varepsilon > 0$, so that

$$f^2(x) > d^2/2 \quad \text{whenever} \quad x \in [c-\varepsilon, c+\varepsilon]$$



Then:

$$\int_{c-\varepsilon}^{c+\varepsilon} f^2(x)\, dx \geq \int_{c-\varepsilon}^{c+\varepsilon} \frac{d^2}{2}\, dx = \frac{d^2}{2}\overbrace{\left(c+\varepsilon - (c-\varepsilon)\right)}^{2\varepsilon}$$

$$= d^2 \varepsilon > 0$$

Then:

$$\int_a^b f^2(x)\,dx = \underbrace{\int_a^{c-\varepsilon} f^2(x)\,dx}_{\text{b/c } f^2(x) \geq 0} + \boxed{\int_{c-\varepsilon}^{c+\varepsilon} f^2(x)\,dx} + \underbrace{\int_{c+\varepsilon}^{b} f^2(x)\,dx}_{\text{b/c } f^2(x) \geq 0}$$

$$\geq \quad 0 \quad + \quad d^2\varepsilon \quad + \quad 0$$

So:

$$\int_a^b f^2(x)\,dx \geq d^2\varepsilon > 0 .$$

$\square$