7.1 Inner Product Spaces

Generalize the dot product in $\mathbb{R}^n$:

$\vec{u} = \langle u_1, u_2, \ldots, u_n \rangle \quad \vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$

$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$.

Definition (The Axioms of an Inner Product Space):
Let V be a vector space. An ***inner product*** on V is a ***bilinear form*** $\langle\,|\,\rangle$ on V, that is, a ***function*** that takes ***two vectors*** $\vec{u}$, $\vec{v} \in V$, and produces a ***scalar***, denoted $\langle \vec{u} | \vec{v} \rangle$, such that the following properties are satisfied by all vectors $\vec{u}$, $\vec{v}$ and $\vec{w} \in V$:

1. *The Symmetric Property*

$$\langle \vec{u} | \vec{v} \rangle = \langle \vec{v} | \vec{u} \rangle$$

2. *The Additive Property*

$$\langle \vec{u} + \vec{v} | \vec{w} \rangle = \langle \vec{u} | \vec{w} \rangle + \langle \vec{v} | \vec{w} \rangle$$

3. *The Homogenous Property*

$$\langle k \cdot \vec{u} | \vec{v} \rangle = k \cdot \langle \vec{u} | \vec{v} \rangle$$

4. *The Positive Property*

$$\text{If } \vec{v} \neq \vec{\mathbf{0}}_V, \text{ then } \langle \vec{v} | \vec{v} \rangle > 0.$$

We also say that V is an ***inner product space*** under the inner product $\langle\,|\,\rangle$.

How About the Zero Vector?

Theorem: Let V be an inner product space. Then, for any $\vec{v} \in V$:

$$\left\langle \vec{v} | \vec{\mathbf{0}}_V \right\rangle = \left\langle \vec{\mathbf{0}}_V | \vec{v} \right\rangle = 0.$$

In particular:

$$\left\langle \vec{\mathbf{0}}_V | \vec{\mathbf{0}}_V \right\rangle = 0.$$

<u>Idea</u>

$$\langle \vec{v}, \vec{0} + \vec{0} \rangle = \langle \vec{v}, \vec{0} \rangle$$

$$\parallel$$

$$\langle \vec{v}, \vec{0} \rangle + \langle \vec{v}, \vec{0} \rangle = \langle \vec{v}, \vec{0} \rangle.$$

Other Easy Consequences

A1

$= \langle k\vec{v}, \vec{u}\rangle$

A3

$= k\langle \vec{v}, \vec{u}\rangle$

$= k\langle \vec{u}, \vec{v}\rangle$

A1

$$\langle \vec{u}\,|\,k \bullet \vec{v}\rangle = k \bullet \langle \vec{u}\,|\,\vec{v}\rangle, \quad \text{and}$$

$$\langle \vec{u}\,|\,\vec{v} + \vec{w}\rangle = \langle \vec{u}\,|\,\vec{v}\rangle + \langle \vec{u}\,|\,\vec{w}\rangle$$

If $k = -1$, we also have:

$$\langle \vec{u} - \vec{v}\,|\,\vec{w}\rangle = \langle \vec{u} + (-1 \bullet \vec{v})\,|\,\vec{w}\rangle = \langle \vec{u}\,|\,\vec{w}\rangle + \langle -1 \bullet \vec{v}\,|\,\vec{w}\rangle$$

$$= \langle \vec{u}\,|\,\vec{w}\rangle - \langle \vec{v}\,|\,\vec{w}\rangle \quad \text{and}$$

$$\langle \vec{u}\,|\,\vec{v} - \vec{w}\rangle = \langle \vec{u}\,|\,\vec{v}\rangle - \langle \vec{u}\,|\,\vec{w}\rangle.$$

Weighted Dot Products

Let $\vec{u}$ and $\vec{v}$ be members of $\mathbb{R}^n$, and

let $\gamma_1, \ \gamma_2 \ . \ . \ . \ \gamma_n$ be ***positive*** real numbers.

Define:

$$\langle \vec{u} \,|\, \vec{v} \rangle = \overbrace{\gamma_1 u_1 v_1 + \gamma_2 u_2 v_2 + \cdots + \gamma_n u_n v_n}^{\in \mathbb{R}}$$

Symmetry ✓

Add ✓

Hom ✓

dot

Add $\langle \vec{u} + \vec{v}, \vec{w} \rangle = [\gamma_1 (u_1 + v_1) w_1] + \cdots\cdots + [\gamma_n (u_n + v_n) w_n]$

$= [\gamma_1 u_1 w_1 + \gamma_1 v_1 w_1] + \cdots\cdots + [\gamma_n u_n w_n + \gamma_n v_n w_n]$

$=$ a lot of regrouping

$= [\gamma_1 u_1 w_1 + \cdots\cdots + \gamma_n u_n w_n] + [\quad]$

$= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$.

Inner Products Generated by Isomorphisms

We can generalize the dot product in $\mathbb{R}^n$ further by considering any ***isomorphism***:

$T(\vec{u}) \in \mathbb{R}^n$

$$\boxed{T : \mathbb{R}^n \to \mathbb{R}^n}$$

1-1, onto
$[T]$ invertible

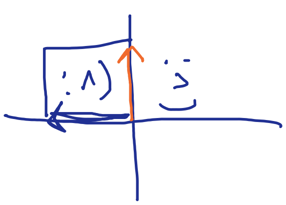
(that is, a one-to-one and onto operator) and define a new inner product on $\mathbb{R}^n$ by:

usual dot product!

$$\langle \vec{u} | \vec{v} \rangle_T = \underbrace{T(\vec{u})}_{\in \mathbb{R}^n} \circ \underbrace{T(\vec{v})}_{\in \mathbb{R}^n}$$

Ex $[T] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ rotation by $90°$ CCW.

invertible? $\det([T]) = 0 \cdot 0 - 1(-1) = 1 \neq 0$.



$\vec{u} = \langle 2, 3 \rangle$
$\vec{v} = \langle -4, 1 \rangle$

$$\langle \vec{u}, \vec{v} \rangle_T = [T(\vec{u})] \circ [T(\vec{v})]$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} \circ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 2 \end{bmatrix} \circ \begin{bmatrix} -1 \\ -4 \end{bmatrix} = \langle -3, 2 \rangle \circ \langle -1, -4 \rangle$$

$$= (-3)(-1) + (2)(-4) = 3 - 8 = \boxed{-5}$$

Polynomial Evaluations

Let $p(x)$ and $q(x)$ be members of $\mathbb{P}^n$, and

let $c_1,\ c_2, \ldots,\ c_n,\ c_{n+1}$ be ***any*** real numbers.

Define:

$$\langle p(x) \,|\, q(x) \rangle = p(c_1)q(c_1) + p(c_2)q(c_2) + \cdots + p(c_n)q(c_n) + p(c_{n+1})q(c_{n+1}).$$

Inner Products Induced by Integrals

Consider $C(I)$, the vector space of all ***continuous*** functions on $I = [a, b]$.

$f, g \in \mathcal{C}([a,b], \mathbb{R})$ continuous

Define:

$$\boxed{\langle f(x) \,|\, g(x) \rangle = \int_a^b \underbrace{f(x) \cdot g(x)}_{\text{cont.}} \, dx}$$

Thm if f cont, Riemann integral $\int_a^b f$ exists.

This appears in Math 55 when constructing ***Fourier Series***.

Ex $\mathcal{C}([0, \pi/2], \mathbb{R})$. Compute $\langle f, g \rangle$ if $f(x) = \sin(x)$
$g(x) = \cos(x)$

$$\langle \sin(x), g(x) \rangle = \int_0^{\pi/2} \sin(x)\cos(x)\,dx = \frac{(\sin(x))^2}{2}\Big|_0^{\pi/2} = \frac{1}{2} - \frac{0}{0} = \boxed{\frac{1}{2}}.$$

$u = \sin(x)$
$du = \cos(x)\,dx$

$\mathcal{C}([0, 2\pi], \mathbb{R})$.

$$\langle \sin(x), \cos(x) \rangle = \int_0^{2\pi} \sin(x)\cos(x)\,dx = \frac{(\sin(x))^2}{2}\Big|_0^{2\pi} = \boxed{0}.$$

A Non-Example

(Non-)Example: Let $\mathbb{R}^2$ be given the bilinear form:

$$\langle \vec{u} | \vec{v} \rangle = u_1 v_2 + u_2 v_1$$