7.2 Geometric Constructions using Inner Products

V, vector space. $\langle \cdot, \cdot \rangle$ an inner product on V.

$(V, \langle \cdot, \cdot \rangle)$ inner product space. (IPS).

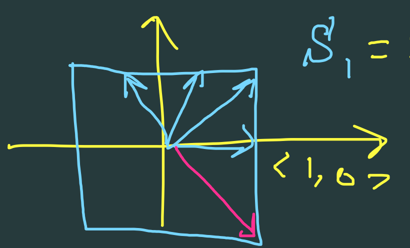
def norm $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$ norm of $\vec{v}$

unit sphere $S_1 = \{ \vec{v} \in V \mid \|\vec{v}\| = 1 \}$ in name only!

Ex $V = \mathbb{R}^2$, ~~$\langle \vec{v}, \vec{w} \rangle = \max(|v_1 - w_1|, |v_2 - w_2|)$~~

$\vec{v} = \langle v_1, v_2 \rangle$ $\|\vec{v}\| = \max(|v_1|, |v_2|)$

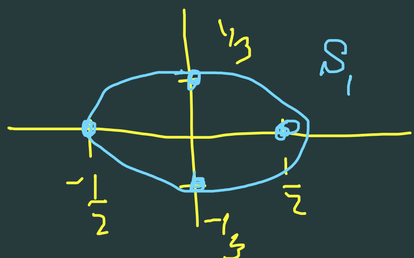
$\vec{w} = \langle w_1, w_2 \rangle$



$S_1 = \{ \vec{v} \in \mathbb{R}^2 \mid \|\vec{v}\| = 1 \}$

Ex $V = \mathbb{R}^2$, $\langle \vec{v}, \vec{w} \rangle = 4 v_1 w_1 + 9 v_2 w_2$ (weighted dot product)

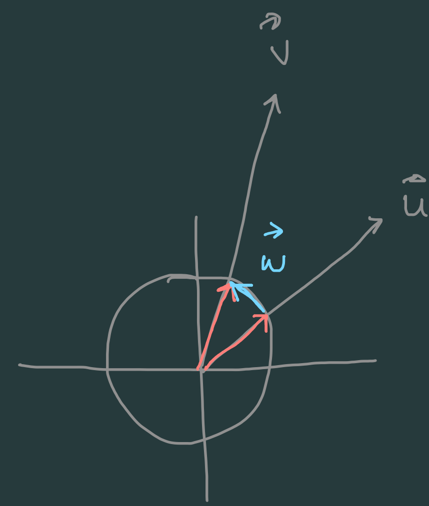
$\|\vec{v}\| = 4v_1^2 + 9v_2^2 = 1$ $\qquad 4x^2 + 9y^2 = 1$



Thm Cauchy-Schwarz Inequality

↳ generalization of Law of Cosines.

$$\boxed{\langle \vec{v}, \vec{w} \rangle \leq \|\vec{v}\| \cdot \|\vec{w}\|}$$



Pf Exercise. Same as proof in $\mathbb{R}^n$.

Case $\vec{u} \neq \vec{0}$, $\vec{v} = \vec{0}$. ✓

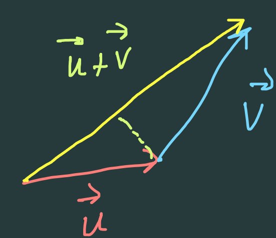
Case $\vec{u} \neq \vec{0}$, $v \neq \vec{0}$. Let $\vec{w} = \dfrac{\vec{u}}{\|\vec{u}\|} - \dfrac{\vec{v}}{\|\vec{v}\|}$

Compute $\langle \vec{w}, \vec{w} \rangle \geq 0$ start b/c positivity

$\square$

Thm Triangle Inequality: $\forall \vec{u}, \vec{v} \in V$ IPS

$$\boxed{\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|}$$



Pf Exercise. $\square$ Great test Q!

def Angles $\boxed{\theta = \cos^{-1}\left(\dfrac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \, \|\vec{v}\|} \right)}$

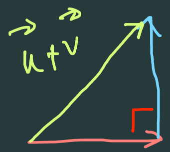
☆ Orthogonal ☆: $\vec{u}$ & $\vec{v}$ are orthogonal vectors in V

iff $\langle \vec{u}, \vec{v} \rangle = 0$.

$\boxed{\text{Thm}}$ Generalized Pythagorean Theorem

If $\vec{u}$ & $\vec{v}$ are orthogonal vectors in V:

then $$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$$



Pf $\|\vec{u}+\vec{v}\|^2 = \langle u+v, u+v \rangle \ldots$ $\square$