7.3 Gram-Schmidt Algorithm

7_29_2019

Motivation In $\mathbb{R}^n$, special basis: $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ basis = LI & Span

special: $\vec{e}_1 \circ \vec{e}_2 = \langle 1,0,0,\dots,0\rangle \circ \langle 0,1,0,0,\dots,0\rangle$

$= (1)(0) + (0)(1) + (0)(0) + \dots + (0)(0) = 0.$

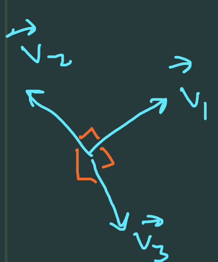
$\|\vec{e}_i\| = 1$ $\quad \vec{e}_i \circ \vec{e}_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$ say $\{e_1, \dots, e_n\}$ is orthonormal basis

↳ orthogonal ↳ length = 1

Goal Start w/ a basis $B = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n\}$ for V, $\dim(V) = n$.

End w/ a orthonormal basis $\tilde{B} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$

def $\tilde{B}$ is an orthonormal basis if:

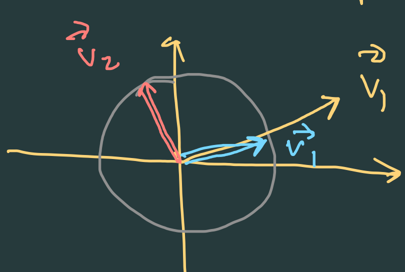


① orthogonal condition: $\langle \vec{v}_i, \vec{v}_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ \neq 0 & \text{if } i = j \end{cases}$

② normal condition: $\|\vec{v}_i\| = 1$ for all $i = 1, 2, \dots, n$

A set B is called orthogonal if it satisfies just condition ①.

Example Warmup: all intuition comes from $\mathbb{R}^2$:



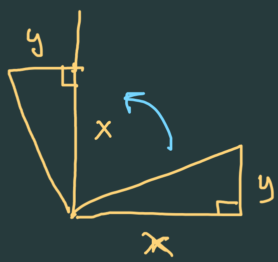
Build O.N. basis for $\mathbb{R}^2$:

$\vec{v}_1$ = any non-zero vector.

$\vec{v}_1$ = any unit length vector. $\quad \vec{v}_1 = \langle x, y \rangle$

condition: $x^2 + y^2 = 1$

$\vec{v}_2 = \langle -y, x \rangle$ so $\{v_1, v_2\}$ is an O.N. basis



- $\vec{v}_1 \circ \vec{v}_2 = \langle x, y \rangle \circ \langle -y, x \rangle$

$= (x)(-y) + (y)(x) = 0.$

☆ **Thm** ☆ Let $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is orthogonal set.

Then S is linearly independent.

Pf Excercise. □ Good test ☺!

Let $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ O.N. basis for V

↳ LI, & Span

Thm Let $\vec{v} \in V$.

$$\boxed{\langle \vec{v} \rangle_B = \langle c_1, c_2, \dots, c_n \rangle} \quad \& \quad \boxed{c_i = \langle \vec{v}, \vec{v}_i \rangle}$$

Pf $\vec{v} \in V = \operatorname{Span}(\{\vec{v}_1, \dots, \vec{v}_n\})$ so $\exists\ c_1, c_2, \dots, c_n \in \mathbb{R}$:

$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$. ☆ Important Techique ☆

- Compute: $\langle \vec{v}, v_1 \rangle = \langle c_1\vec{v}_1 + \dots + c_n\vec{v}_n, \vec{v}_1 \rangle$

$$= c_1\underbrace{\langle \vec{v}_1, \vec{v}_1 \rangle}_{\neq 0} + \underbrace{c_2\underbrace{\langle \vec{v}_2, \vec{v}_1 \rangle}_{=0} + \dots + c_n\underbrace{\langle \vec{v}_n, \vec{v}_1 \rangle}_{=0}}_{=0}$$

so $\langle \vec{v}, \vec{v}_1 \rangle = c_1 \underbrace{\langle v_1, v_1 \rangle}_{=1}$

so $\langle \vec{v}, \vec{v}_1 \rangle = c_1$. True $i=1$.

Similarly:

$\langle \vec{v}, \vec{v}_2 \rangle = \dots\dots = c_2$.

$\langle \vec{v}, \vec{v}_i \rangle = c_i$. □

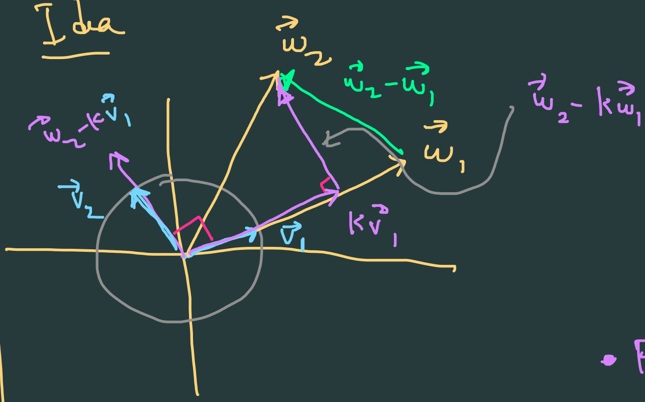
Gram-Schmidt Algorithm V IPS, $\dim(V) = n$.

start $B = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n\}$ basis for V. $\mathrm{Span}(B) = V$, B LI

end $\tilde{B} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ orthonormal basis for V

- LI
- $\mathrm{Span}(\tilde{B}) = \mathrm{Span}(B) = V$
- orgonomal normal : $\langle \vec{v}_i, \vec{v}_j \rangle = \begin{cases} 1 & \text{if } i=j \text{ (unit length)} \\ 0 & \text{if } i \neq j \text{ (orthogonal)} \end{cases}$

Idea



- find the correct scalar k so that

$$\vec{w}_2 - k\vec{v}_1 \perp \vec{v}_1$$

- $\vec{w}_2 - k\vec{v}_1$ is right "direction" of $\vec{v}_2$

so we just make it unit length.

- Finding k:

want $\langle \vec{w}_2 - k\vec{v}_1, \vec{v}_1 \rangle = 0$

iff $\langle \vec{w}_2, \vec{v}_1 \rangle - k \underbrace{\langle \vec{v}_1, \vec{v}_1 \rangle}_{=1} = 0$ iff $\dfrac{\langle \vec{w}_2, \vec{v}_1 \rangle}{\langle v_1, v_1 \rangle} = k$!

Gram-Schmidt Algorithm *Modified*

Start $B = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n\}$ basis. **Intermediate**: $B' = \{\vec{v}_1, \dots, \vec{v}_n\}$ is orthogonal. NOT length 1.

END $\tilde{B}$ = divide vectors in B' to make them unit length.

Step 1 $\boxed{\vec{v}_1 = \vec{w}_1}$

- If $\dim V = 1$, done! Since $\operatorname{Span}(\{\vec{v}_1\}) = \operatorname{Span}(\{\vec{w}_1\}) = V$, since $\vec{w}_1 \neq \vec{0}$. $\vec{v}_1 \neq \vec{0}$
- Otherwise, continue:

Step 2 $\boxed{\vec{v}_2 = \vec{w}_2 - \dfrac{\langle \vec{w}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1}$

- If $\dim(V) = 2$, done!
- **Check** $\langle \vec{v}_2, \vec{v}_1 \rangle = 0$:

$$\Big\langle w_2 - \Big(\underbrace{\frac{\langle w_2, v_1\rangle}{\langle v_1, v_1\rangle}}_{\text{scalar}}\Big) v_1, v_1 \Big\rangle$$

$$= \langle w_2, v_1 \rangle - \left(\frac{\langle w_2, v_1\rangle}{\cancel{\langle v_1, v_1\rangle}}\right) \cancel{\langle v_1, v_1 \rangle}$$

$$= \langle w_2, v_1 \rangle - \langle w_2, v_1 \rangle = 0 \checkmark$$

- **Check** $\underbrace{\operatorname{Span}(\{\vec{v}_1, \vec{v}_2\})}_{X} = \underbrace{\operatorname{Span}(\{\vec{w}_1, \vec{w}_2\})}_{Y}$
  - $\vec{v}_2 \in Y$ & $\vec{v}_1 \in Y$ so $\{v_1, v_2\} \subset Y$ & v_1, v_2 is orthogonal $\Rightarrow$ LI.

    so $\operatorname{Span}(\{v_1, v_2\}) = Y$.

Step 3 $\boxed{\vec{v}_3 = \vec{w}_3 - \dfrac{\langle \vec{w}_3, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 - \dfrac{\langle \vec{w}_3, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle} \vec{v}_2}$

If $\dim(V) = 3$, done. Otherwise continue.

- **check** $\langle \vec{v}_3, \vec{v}_1 \rangle = 0$ & $\langle \vec{v}_3, \vec{v}_2 \rangle = 0$

$$\langle \vec{v}_3, \vec{v}_2 \rangle = \Big\langle w_3 - \left(\frac{\langle w_3, v_1\rangle}{\langle v_1, v_1\rangle}\right)\vec{v}_1 - \left(\frac{\langle w_3, v_2\rangle}{\langle v_2, v_2\rangle}\right) v_2, \vec{v}_2 \Big\rangle$$

$$= \langle w_3, v_2 \rangle - \left(\frac{\langle w_3, v_1\rangle}{\langle v_1, v_1\rangle}\right) \underbrace{\langle v_1, v_2 \rangle}_{=0} - \left(\frac{\langle w_3, v_2\rangle}{\cancel{\langle v_2, v_2\rangle}}\right) \cancel{\langle v_2, v_2 \rangle} = \langle w_3, v_2 \rangle - \langle w_3, v_2 \rangle = 0$$

- **Check** $\underbrace{\operatorname{Span}(\{v_1, v_2, v_3\})}_{X} = \underbrace{\operatorname{Span}(\{w_1, w_2, w_3\})}_{Y}$
  - $\vec{v}_1 \in Y$, $\vec{v}_2 \in Y$
  - formula for $\vec{v}_3$ says its LC of $\vec{w}_1, \vec{w}_2, \vec{w}_3$ b/c v_1, v_2 are LC of these. so $\vec{v}_3 \in Y$.

so $\{v_1, v_2, v_3\} \in Y$ & orthogonal set, so LI.

⋮

Step k+1 Assume $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ are already constructed & Orthog. set.

& $\text{Span}(\{\vec{v}_1, \dots, \vec{v}_k\}) = \text{Span}(\{\vec{w}_1, \dots, \vec{w}_k\})$

$$\boxed{\vec{v}_{k+1} = \vec{w}_{k+1} - \frac{\langle \vec{w}_{k+1}, \vec{v}_1 \rangle}{\langle v_1, v_1 \rangle} \vec{v}_1 - \frac{\langle \vec{w}_{k+1}, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle} \vec{v}_2 - \cdots - \frac{\langle \vec{w}_{k+1}, \vec{v}_k \rangle}{\langle \vec{v}_k, \vec{v}_k \rangle} \vec{v}_k}$$

- Check $\langle \vec{v}_{k+1}, \vec{v}_i \rangle = 0$, $i = 1, \dots, k$.
- check $\text{Span}(\{\vec{v}_1, \dots, \vec{v}_{k+1}\}) = \text{Span}(\{\vec{w}_1, \dots, \vec{w}_{k+1}\})$.

- At some point this process must end!

  why? B/c $\dim(V) = n$

At this point: $\{\vec{v}_1, \dots, \vec{v}_n\}$ constructed is LI, orthogonal but not unit length! This constructs B'.

- At each step, can do helpful simplification:

  can replace each $\vec{v}_i$ w/ a parallel of it with NO fractions & NO radicals

FINAL STEP $\tilde{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ w/ replace $\vec{v}_i$ from B' w/ $\dfrac{\vec{v}_i}{\|\vec{v}_i\|}$

at this final step, $\tilde{B}$ is all unit length vectors!

Example $\mathbb{R}^4$, usual dot product. Let B be a basis. Use Gram-Schmidt to find an O.N. basis $\tilde{B}$

$$B = \{ \vec{w}_1 = \langle 1,-1,1,0\rangle, \quad \vec{w}_2 = \langle 1,0,1,-1\rangle, \quad \vec{w}_3 = \langle -1,1,1,1\rangle, \quad \vec{w}_4 = \langle 1,1,1,-1\rangle \}$$

Step 1 $\vec{v}_1 = \vec{w}_1$ $\boxed{\vec{v}_1 = \langle 1,-1,1,0\rangle}$ Note $\|v_1\|^2 = v_1 \circ v_1 = 3$ $\boxed{\|\vec{v}_1\| = \sqrt{3}}$

Step 2 $\vec{v}_2 = \vec{w}_2 - \dfrac{\vec{w}_2 \circ \vec{v}_1}{v_1 \circ v_1}\vec{v}_1 =$

$$= \langle 1,0,1,-1\rangle - \left(\frac{\overbrace{\langle 1,0,1,-1\rangle \circ \langle 1,-1,1,0\rangle}^{1+0+1+0=2}}{\underbrace{\langle 1,-1,1,0\rangle \circ \langle 1,-1,1,0\rangle}_{1+1+1=3}}\right)\langle 1,-1,1,0\rangle = \langle 1,0,1,-1\rangle - \langle \tfrac{2}{3}, \tfrac{-2}{3}, \tfrac{2}{3}, 0\rangle$$

$\vec{v}_2 = \langle \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{3}, -1\rangle$ $\boxed{\vec{v}_2 = \langle 1,2,1,-3\rangle}$ Note $\|\vec{v}_2\|^2 = v_2 \circ v_2 = 15$ $\boxed{\|v_2\| = \sqrt{15}}$

Step 3 $\vec{v}_3 = \vec{w}_3 - \dfrac{w_3 \circ v_1}{v_1 \circ v_1} v_1 - \dfrac{w_3 \circ v_2}{v_2 \circ v_2} v_2$

$$= \langle -1,1,1,1\rangle - \left(\frac{\overbrace{\langle -1,1,1,1\rangle \circ \langle 1,-1,1,0\rangle}^{-1-1+1+0=-1}}{3}\right)\langle 1,-1,1,0\rangle - \left(\frac{\overbrace{\langle -1,1,1,1\rangle \circ \langle 1,2,1,-3\rangle}^{-1+2+1-3=-1}}{\underbrace{\langle 1,2,1,-3\rangle \circ \langle 1,2,1,-3\rangle}_{1+4+1+9=15}}\right)\langle 1,2,1,-3\rangle$$

$$= \langle -1,1,1,1\rangle - \left(\tfrac{-1}{3}\right)\langle 1,-1,1,0\rangle - \left(\tfrac{-1}{15}\right)\langle 1,2,1,-3\rangle$$

$$= \langle [-1+\tfrac{1}{3}+\tfrac{1}{15}], [1-\tfrac{1}{3}+\tfrac{1}{15}], [1+\tfrac{1}{3}+\tfrac{1}{15}], [1+0-\tfrac{3}{15}]\rangle$$

$$= \langle -\tfrac{9}{15}, \tfrac{12}{15}, \tfrac{21}{15}, \tfrac{12}{15}\rangle = \frac{\langle -3,4,7,4\rangle}{5}$$

$\boxed{\vec{v}_3 = \langle -3,4,7,4\rangle}$ $\|v_3\|^2 = \langle v_3, v_3\rangle = 90$ $\boxed{\|\vec{v}_3\| = \sqrt{90}}$

Step 4 $\vec{v}_4 = w_4 - \left(\frac{w_4 \circ v_1}{\underbrace{v_1 \circ v_1}_{3}}\right) v_1 - \left(\frac{w_4 \circ v_2}{\underbrace{v_2 \circ v_2}_{15}}\right) v_2 - \left(\frac{w_4 \circ v_3}{v_3 \circ v_3}\right) v_3$

$$= \langle 1,1,1,-1\rangle - \left(\frac{\overset{1}{\langle 1,1,1,-1\rangle \circ \langle 1,-1,1,0\rangle}}{3}\right)\langle 1,-1,1,0\rangle$$

$$- \left(\frac{\overset{7}{\langle 1,1,1,-1\rangle \circ \langle 1,2,1,-3\rangle}}{15}\right)\langle 1,2,1,-3\rangle$$

$$- \left(\frac{\overset{4}{\langle 1,1,1,-1\rangle \circ \langle -3,4,7,4\rangle}}{\langle -3,4,7,4\rangle \circ \langle -3,4,7,4\rangle}\right)\langle -3,4,7,4\rangle$$

$9+16+49+16 = 90$

$$= \langle 1,1,1,-1\rangle - \frac{1}{3}\langle 1,-1,1,0\rangle - \frac{7}{15}\langle 1,2,1,-3\rangle - \frac{4}{90}\langle -3,4,7,4\rangle$$

$$= \left\langle \frac{15}{45}, \frac{10}{45}, \frac{-5}{45}, \frac{10}{45}\right\rangle = \left\langle \frac{1}{3}, \frac{2}{9}, \frac{-1}{9}, \frac{2}{9}\right\rangle$$

$\boxed{\vec{v}_4 = \langle 3,2,-1,2\rangle}$ $\quad \|\vec{v}_4\| = \sqrt{9+4+1+4} = \sqrt{18}$ $\quad \boxed{\|\vec{v}_4\| = \sqrt{18}}$

- orthogonal set $B' = \{\langle 1,-1,1,0\rangle, \langle 1,2,1,-3\rangle, \langle -3,4,7,4\rangle, \langle 3,2,-1,2\rangle\}$
- orthonormal set $\tilde{B} = \left\{\frac{1}{\sqrt{3}}\langle 1,-1,1,0\rangle, \frac{1}{\sqrt{15}}\langle 1,2,1,-3\rangle, \frac{1}{\sqrt{90}}\langle -3,4,7,4\rangle, \frac{1}{\sqrt{18}}\langle 3,2,-1,2\rangle\right\}$

Ex Let $\vec{v} = \langle 4,2,-1,3\rangle \in \mathbb{R}^4$. Find $\langle \vec{v}\rangle_{\tilde{B}} = \langle c_1, c_2, c_3, c_4\rangle_{\tilde{B}}$

$$\langle 4,2,-1,3\rangle = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4$$

$$\begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{array}$$
$$\left[\begin{array}{cccc|c} 1/\sqrt{3} & 1/\sqrt{15} & -3/\sqrt{90} & 3/\sqrt{18} & 4 \\ -1/\sqrt{3} & 2/\sqrt{15} & 4/\sqrt{90} & 2/\sqrt{18} & 2 \\ 1/\sqrt{3} & 1/\sqrt{15} & 7/\sqrt{90} & -1/\sqrt{18} & -1 \\ 0 & -3/\sqrt{15} & 4/\sqrt{90} & 2/\sqrt{18} & 3 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & c_1 \\ 0 & 1 & 0 & 0 & c_2 \\ 0 & 0 & 1 & 0 & c_3 \\ 0 & 0 & 0 & 1 & c_4 \end{array}\right]$$

Ex $W = \text{Span}(\{\vec{w}_1, \vec{w}_2\})$ $\qquad W^\perp$ o.n. basis

Find o.N. basis for W $\quad \{\vec{v}_1, \vec{v}_2\}$ of $\tilde{B}$ $\qquad \{\vec{v}_3, \vec{v}_4\}$

Ex Find orthogonal decomposition:

$$W + W^{\perp} = \mathbb{R}^4 \qquad \vec{v} = \langle a_1, a_2, a_3, a_4 \rangle$$

$$\vec{v} = \vec{w}_1 + \vec{w}_2 \quad \text{where} \quad \vec{w}_1 \in W, \quad \vec{w}_2 \in W^{\perp}$$

$$\langle c_1, c_2, c_3, c_4 \rangle = \underbrace{\left(c_1\vec{v}_1 + c_2\vec{v}_2 \right)}_{\in W} + \underbrace{\left(c_3\vec{v}_3 + c_4\vec{v}_4 \right)}_{W^{\perp}}$$