What is Linear Algebra?

A Bird's Eye View

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Outline

What is Linear Algebra? Beauty & Importance

- A beautiful subject . . . why?
  - real mathematical theory
  - (likely) your first love (er...exposure) of proofs
  - moves effortlessly from lines, to planes, to hyperplanes, to n-dimensional space $\mathbb{R}^n$
  - you'll learn to "see" 12-dimensional space
  - Ditch geometry, but don't ditch geometry!
  - theory with purpose! So many APPs (applications)
- Enormous importance! Maybe even greater than Calculus!

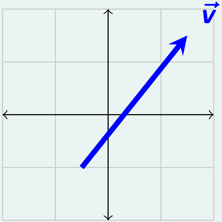
What is Linear Algebra? Beauty & Importance

- LAPs (Linear Algebra Applications):
  - Economics: Leontief model of Economics (1950s Harvard professor $\rightarrow$ won Nobel prize in Econ)
  - Physics: so many! A few are. . .
    - vectors in 2D, 3D, and higher dimensions
    - Forces, Electrice Fields, Magnetic Fields, . . .
    - Quantum Mechanics (uses ∞-dimensional LA)
  - Data Science: stats + LA + Calc + programming
  - Engineering:
  - MATH duh!
    - most (all?) mathematical courses use LA in some way
- But . . . we'll not study any of these applications in detail INSTEAD we'll lay the foundations for these APPs

What is Linear Algebra? . . . something to do with vectors?

We start with vectors. Vectors according to. . .

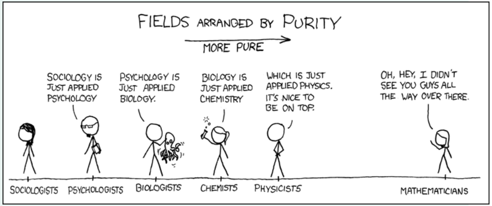
Physicists
"*something* with a magnitude & direction"



Computer Scientists
"a list (or array) of numbers"

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ -5 \end{bmatrix}$$

Mathematicians
"an element of a vector space"



What is Linear Algebra?

Dr. Basilio

Outline

Beauty & Importance

Vectors

Systems of

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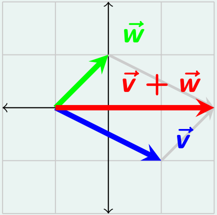
inear Algebra?

Properties of vectors

Physicists

"*something* with a magnitude & direction"



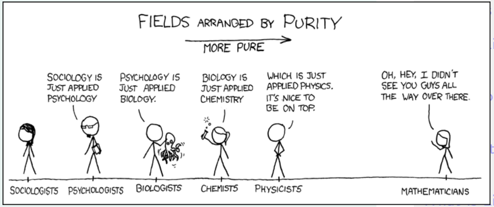
Computer Scientists

"a list (or array) of numbers"

$$\vec{v} + \vec{w} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ -5 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 2 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \\ -2 \\ -1 \end{bmatrix}$$

Mathematicians

"an element of a vector space"



What is Linear Algebra? . . . something to do with vectors?

Mathematician's view abstracts the properties shared by many different objects studied over a long period of time. So, mathematician's care **only** about structure of objects not superficially what they look like.

- Vectors live in vector spaces. Vector spaces are collections of objects that satisfy many properties. The most important are:

- $\mathbb{Z}^n$
- $\mathbb{Q}^n$
- $\mathbb{R}^n$
- $\mathbb{C}^n$
- $\mathbb{M}_{n\times n}$ - space of all $n \times n$ matrices
- $P(\mathbb{R}, \mathbb{R})$ - space of all polynomials
- $C([a, b], \mathbb{R})$ - space of all continuous functions
- $C^\infty([a, b], \mathbb{R})$ - space of all differentiable functions
- $\ell_\infty(\mathbb{R})$ - space of all sequences
- space of all power series

What is Linear Algebra? . . . something to do with vectors?

Mathematician's care **only** about structure of objects not superficially what they look like.

- What do these examples have in common?
- **Addition:** there is a "natural" way to define how to add two objects
- **Scalar Multiplication:** there is a "natural" what to define what multiplying an object by a real number α
- Example: using computer scientist's concept of vector, we can define "addition" and "scalar multiplication" via components

$$\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \\ -2 \end{bmatrix} \quad \text{and} \quad \alpha \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4\alpha \\ 2\alpha \\ \alpha \\ -2\alpha \end{bmatrix}$$

What is Linear Algebra? ... matrices?

Mathematician's care **only** about structure of objects not superficially what they look like.

- But isn't linear algebra $\Longleftrightarrow$ Matrix Algebra?
- **Addition:** there is a "natural" way to define how to add two objects
- **Scalar Multiplication:** there is a "natural" what to define what multiplying an object by a real number α
- Example: For matrices we define "addition" and "scalar multiplication" via components as well

$$\begin{bmatrix} 1 & -7 \\ 2 & 8 \\ -1 & -3 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 2 & -5 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 2 & -1 \\ 1 & -8 \\ -2 & 8 \end{bmatrix} \text{ and } \alpha \cdot \begin{bmatrix} 4 & 6 \\ 2 & -1 \\ 1 & -3 \\ -2 & 8 \end{bmatrix} = \begin{bmatrix} 4\alpha & 6\alpha \\ 2\alpha & -\alpha \\ \alpha & -3\alpha \\ -2\alpha & 8\alpha \end{bmatrix}$$

What is Linear Algebra? . . . Systems of Equations

In many ways, all of linear algebra boils down to solving a system of equations.

"row picture"

- $$\begin{cases} x + 2y + 3z &= 4 \\ -x + y - 5z &= 0 \\ 2x - y - z &= -1 \end{cases} \iff \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & -5 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$$

"column picture"

$$\iff x \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + z \begin{bmatrix} 3 \\ -5 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$$

What is Linear Algebra? ... Systems of Equations

System of equations. A simple example.

- $\begin{cases} x - y &= -2 \\ x + 2y &= 1 \end{cases} \iff \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \iff \boxed{A\vec{x} = \vec{b}}$
- "row picture" $\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ intersection of two lines
- "column picture" $x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ linear combinations

System of equations. A simple example.

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- "column picture" $x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ linear combinations

Geometry of linear combinations. . .

System of equations.

- Okay, that was easy. So what?
- We need to solve LARGE systems.
- A system with n variables and n unknowns:
- $$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad \qquad \vdots = \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n \end{cases} \Leftrightarrow \boxed{A\vec{x} = \vec{b}}$$

$$\Longleftrightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

- Note: $n = 1000$ is considered "small" nowadays

What is Linear Algebra? ... Systems of Equations

System of equations.

- A system with n variables and n unknowns:
- "Row picture" $$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

  Row picture = intersection of n hyperplanes
- "Column picture" $$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

  Row picture = what linear combination of columns equals $\vec{b}$?

Recall the simple example.

- $\begin{cases} x - y &= -2 \\ x + 2y &= 1 \end{cases} \iff \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \iff \boxed{A\vec{x} = \vec{b}}$
- "column picture" $x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ linear combinations
- Notation: Let $\vec{v}$ =Col 1= $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{w}$ =Col 2= $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$
- Recall: we found that $(-1)\vec{v} + (1)\vec{w} = \vec{b}$ (i.e. $x = -1$, $y = 1$).

The birth of linear algebra is to consider ALL possible linear combinations of $\vec{v}$ and $\vec{w}$!!!!

That is, $\{x_1\vec{v} + x_2\vec{w} \mid x_1, x_2 \in \mathbb{R}\} = \text{span}(\vec{v}, \vec{w})$

Birth of Linear Algebra

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- Notation: Let $\vec{v}$ =Col 1= $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{w}$ =Col 2= $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} a \\ b \end{bmatrix}$
- So, we can write $A = [\, \vec{v} \mid \vec{w} \,]$
- Can we solve this for any $a, b \in \mathbb{R}$?
- YES!
- Why? Because all linear combinations of $\vec{v}$ and $\vec{w}$ will fill the entire plane!!!
- Higher dimensions is much more interesting :-)

Birth of Linear Algebra

To summarize:

- A major problem in linear algebra is to solve a system of equations:

$$\boxed{A\vec{x} = \vec{b}}$$

- This is the same thing as asking: when is $\vec{b}$ a linear combination of the columns vectors of A (i.e. in the span)?
- This is solved using Gauss-Jordan elimination. A clever algorithm that's embarrassingly simple (in principle)

Birth of Linear Algebra

- Next, we generalize our situation to systems of equations with an uneven number of equations and unknowns.
- If we let $m = \#$ equations and $n = \#$ unknowns, we'd like to study
- $$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ \quad \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad = \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m \end{cases} \iff \boxed{A\vec{x} = \vec{b}}$$
- New phenomena can occur now.
- **Imagine** What if we have two 5-dimensional vectors $\vec{v}$ and $\vec{w}$. Can all their linear combinations fill-up all of 5-dimensional space?
- NO! There's too few vectors
- What about five, 5-dimensional vectors? It depends....they must all live in their separate planes. . .

Birth of Linear Algebra

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- **Imagine** What if we have two 5-dimensional vectors $\vec{v}$ and $\vec{w}$. Can all their linear combinations fill-up all of 5-dimensional space?
- NO! There's too few vectors
- What about five, 5-dimensional vectors? It depends....they must all live in their separate planes. . .

- This introduces the important idea of independence. That is, we say a collection of vectors are independent if they "fill up" space as much as possible (a more precise definition will be given later).
- Remarkably, all this information is encoded in the matrix A associated to the SOE.

Birth of Linear Algebra

- Remarkably, all this information is encoded in the matrix A associated to the SOE.
- By studying the structure of A, we can answer many fundamental questions related to a SOE.
- There's four fundamental "subspaces" associated to A
  - Column Space (all linear combinations of columns of A)
  - Row Space (all linear combinations of rows of A)
  - Null space of A
  - Null space of A^T

Gauss-Jordan Elimination

- Meanwhile: almost every problem in linear algebra is solved (one way or another with) Gauss-Jordan Elimination (GJE).
- The method of GJE is used for
  - Checking if a list of vectors are independent
  - Solving SOEs: $A\vec{x} = \vec{b}$
  - Checking if a vector $\vec{b}$ is in the span of of a list of other vectors ($=$ space of all linear combinations)
  - Finding the column space of a matrix
  - Finding the row space of a matrix
  - Finding the rank of a matrix
  - Basically everything in Linear Algebra :-)

Back to Mathematician's vectors

Recall: the mathematician's view of "vectors:" objects you can ADD and scalar multiply. We can do that with matrices!

Addition

- $$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \end{bmatrix} =$$
$$\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & a_{n3}+b_{n3} & \cdots & a_{nn}+b_{nn} \end{bmatrix}$$

Back to Mathematician's vectors

Recall: the mathematician's view of "vectors:" objects you can ADD and scalar multiply. We can do that with matrices!

Scalar Multiplication

$$\alpha \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \alpha a_{13} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \alpha a_{n1} & \alpha a_{n2} & \alpha a_{n3} & \cdots & \alpha a_{nn} \end{bmatrix}$$

What is Linear Algebra?

Linear Algebra is. . .

The study of vector spaces, their structure, and the linear transformations that *map* one vector space to another.

What is Linear Algebra?

Linear Algebra is. . .

The study of vector spaces, their structure, and the linear transformations that *map* one vector space to another.

- We've briefly discussed vectors and vector spaces (the set/collection of all vectors).
- Notation: $\vec{v}$ a vector. V the vector space that collects all vectors. We can write $\vec{v} \in V$.
- The structure of V is easy to describe:
  - "closure under addition": if $\vec{v}, \vec{w} \in V$, then $\vec{v} + \vec{w} \in V$
  - "closure under scalar multiplication:" if $\vec{v}, \vec{w} \in V$, and $a, b \in \mathbb{R}$ are arbitrary, then $a\vec{v} + b\vec{w} \in V$
- The second bullet is what "linear structure" means in an abstract sence.

What is Linear Algebra?

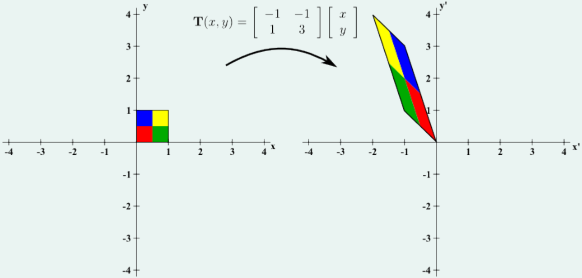
Linear Algebra is...

> The study of vector spaces, their structure, and the linear transformations that *map* one vector space to another.

- The other major goal, once we understand what vector spaces are (and have studied many examples), is to study maps (or functions) between two vector spaces.
- Notation: $L : V \to W$. If $\vec{v} \in V$, then $T(\vec{v}) \in W$
- L because we want to study "linear transformations"
- The structure of L is easy to describe:
  - "additive structure:" if $\vec{v}, \vec{w} \in V$, then $\boxed{T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})}$
  - "multiplicative structure:" if $\vec{v} \in V$ and $a \in \mathbb{R}$, then $\boxed{T(a\vec{v}) = aT(\vec{v})}$

What is Linear Algebra?

- Example in $1D$: $\boxed{L(x) = 2x}$. This is a linear map from $V = \mathbb{R}$ to $W = \mathbb{R}$.
  - L(x+y) = 2(x+y)=2x+2y = L(x)+L(y)
  - L(ax) = 2(ax) = a(2x) = aL(x)
- Example in $2D$: $\boxed{L(\langle x, y\rangle) = \langle -y, x\rangle}$. This is a linear map from $V = \mathbb{R}^2$ to $W = \mathbb{R}^2$ (from $2D$ plane to another $2D$ plane)



What is Linear Algebra?

Linear Algebra is. . .

The study of vector spaces, their structure, and the linear transformations that *map* one vector space to another.

There's so much more to this story!

- Add more structure: inner-products (measure length of vectors and angles). Can do "geometry" on abstract vector spaces.
- & so much more!

Feet back on the ground!

Linear Algebra is. . .

The study of vector spaces, their structure, and the linear transformations that *map* one vector space to another.

Our story begins we the most important vector spaces: $\mathbb{R}^n$:

- $\vec{v} \in \mathbb{R}^n$ is a list of n-tuples:

$$\vec{v} = \langle v_1, v_2, v_3, \ldots, v_n \rangle \qquad \text{or} \qquad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix}$$

Let's begin our journey together :-)