REVIEW OF ALGEBRA, GEOMETRY, FUNCTIONS, & TRIGONOMETRY

Operations : $a, b, c, d \in \mathbb{R}$

- $a(b+c) = ab + ac$
- $a \cdot \left(\frac{b}{c}\right) = \frac{ab}{c}$
- $\frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc}$
- $\frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b}$
- $\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \frac{ad}{bc}$
- $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{ad}$
- $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$
- $\frac{a-b}{c-d} = \frac{b-a}{d-c}$
- $\frac{a-b}{c-d} = -\left(\frac{b-a}{c-d}\right) = -\left(\frac{a-b}{d-c}\right)$
- $\frac{ab+ac}{a} = b+c$

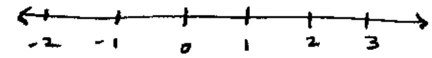
"petals" $\frac{ab+ac}{a} = b+c$

* Expressions in denominators are assumed non-zero *

Exponent Rules : $a, b \in \mathbb{R}$, $m, n \in \mathbb{Q}$

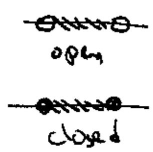
- $a^n \cdot a^m = a^{n+m}$
- $\frac{a^n}{a^m} = a^{n-m}$
- $(a^n)^m = a^{n \cdot m}$
- $a^0 = 1 \quad (a \neq 0)$
- $(a \cdot b)^n = a^n \cdot a^m$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
- $a^{-n} = \frac{1}{a^n}$
- $\frac{1}{a^{-n}} = a^n$
- $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$
- $a^{m/n} = \left(a^{1/n}\right)^m = \left(\sqrt[n]{a}\right)^m$ $(m, n \in \mathbb{Z})$

Real Numbers



$\mathbb{R} = (-\infty, \infty) = \{x \mid -\infty < x < \infty\}$.

Intervals



- open $(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$
- closed $[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$
- other: $[a,b)$, $(a,b]$, $(-\infty, b]$, etc...

Absolute Value: $|a| = \begin{cases} a, & a \geq 0 \\ -a, & a < 0 \end{cases}$

Properties ① $|a \cdot b| = |a| \cdot |b|$ ② $|-a| = |a|$

③ $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ ④ $|a^n| = |a|^n$

Inequalities: $\forall a, b, c, d, k \in \mathbb{R}$:

- $a < b \wedge b < c \Rightarrow a < c$
- $a < b \wedge c < d \Rightarrow a + c < b + d$
- $a < b \Rightarrow a + k < b + k$
- $a < b \wedge k > 0 \Rightarrow a \cdot k < b \cdot k$
- $a < b \wedge k < 0 \Rightarrow a \cdot k > bk$ (notice: ineq. switches!)
- $-|a| \leq a \leq |a|$
- $|a| \leq k \iff -k \leq a \leq k$

  $|a| < k \iff -k < a < k$
- $|a| \geq k \iff a \geq k \vee a \leq -k$ (or)

TRIANGLE INEQ. $\boxed{|a+b| \leq |a| + |b|}$

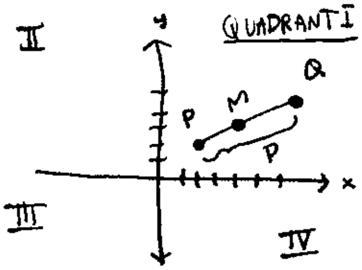
ARITHMETIC-GEOMETRIC MEAN: $\boxed{\sqrt{a \cdot b} \leq \frac{a+b}{2}}$ $a, b > 0$

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$
- Radicals: $n \in \mathbb{N}$, $a \in \mathbb{R}$. A radical is any real number that is a solution to the equation:

$$x^n - a = 0,$$

when it exists. Denoted $\sqrt[n]{a}$ or $a^{1/n}$.

Eg $\sqrt[3]{8} = 2$ since $(\sqrt[3]{8})^3 - 8 = 0$.



$P = (x_1, y_1)$ & $Q = (x_2, y_2)$.

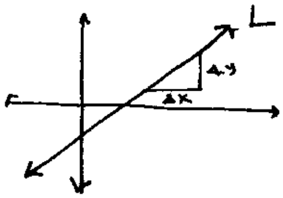
Midpoint $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$

Distance b/w P & Q: $D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

LINES

"constant incline"

"increases (or decreases) by the same rate"



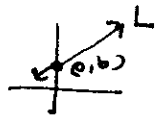
Slope Formula: $\boxed{m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}}$ "rise" / "run"

- Standard Form of Eq of line:

L: $\boxed{ax + by = c}$ (a, b not both zero)

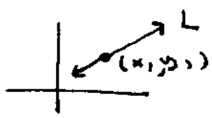
- Slope-Intercept Form:

L: $\boxed{y = mx + b}$ m - slope, $(0, b)$ - y-int



- Point-Slope Form:

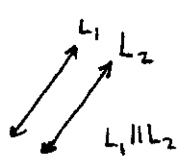
L: $\boxed{y - y_1 = m(x - x_1)}$



- Vertical Lines: $x = c$ $(b = 0, a = 1)$
- Horizontal Lines: $y = c$ $(a = 0, b = 1)$

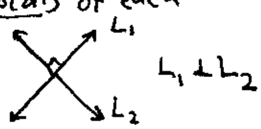
- Parallel lines: two lines with same slope (or both vert. or both horiz)

$\boxed{m_1 = m_2}$



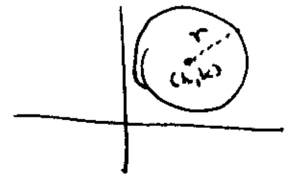
- Perpendicular lines: two lines with slopes are negative reciprocals of each other. (or one vert & one horiz. line)

$\boxed{m_1 = \frac{-1}{m_2}}$



- Important: Horizontal lines have slope 0!

CIRCLES



center: (h, k)
radius: r

Standard Form:

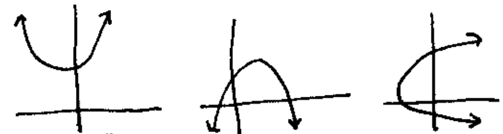
$Ax^2 + Ay^2 + Bx + Cy + D = 0$

Graphing Form: $\boxed{(x - h)^2 + (y - k)^2 = r^2}$

SEMICIRCLES: TOP $y = k + \sqrt{r^2 - (x - h)^2}$

BOTTOM $y = k - \sqrt{r^2 - (x - h)^2}$

PARABOLAS



"functions" "not a function"

Up/Down

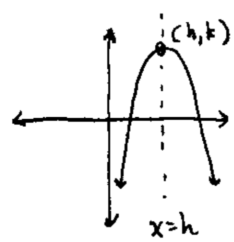
$y = ax^2 + bx + c$ $(a \neq 0)$

Left/Right

$x = ay^2 + by + c$ $(a \neq 0)$

Standard Form $(a \neq 0)$

$$\boxed{\begin{array}{l} y = ax^2 + bx + c \\ y = a(x - h)^2 + k \end{array}}$$



- Vertex: (h, k)
- Axis of Symmetry: $x = h$
- Effect of a:
  - $a > 0 \rightarrow$ opens up
  - $a < 0 \rightarrow$ opens down
  - $|a| > 1 \rightarrow$ thinner than $y = x^2$
  - $|a| < 1 \rightarrow$ wider than $y = x^2$

Vertex Formula

$\boxed{h = \frac{-b}{2a}}$

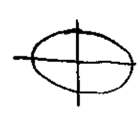
to find k, plug h into formula for y.

$k = f(h) = f\left(\frac{-b}{2a}\right)$

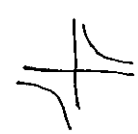
if we set $f(x) = ax^2 + bx + c$.

OTHER

Ellipses: $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$



HYPERBOLAS: $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$



FACTORING

- $a^2 + 2ab + b^2 = (a+b)(a+b)$
- $a^2 - 2ab + b^2 = (a-b)(a-b)$
- $a^2 - b^2 = (a+b)(a-b)$
- $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

SOLVING QUADRATIC EQs

$ax^2 + bx + c = 0$ (QE)

1. try to factor, if applicable
2. Use SRP, if applicable
3. Use complete the Square (CtS) if forced to
4. Use Quadratic Formula:

(QF) $\boxed{x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$

NOTE: this formula gives the one or two solutions to the Quadratic EQ (QE) above.

SRP: Square Root Property: Given $a \in \mathbb{R}$

Solutions to $x^2 = a$ are $x = \pm\sqrt{a}$ but only for $a \geq 0$.

Complete the Square: CtS

$$x^2 + Bx = \left(x + \frac{B}{2}\right)^2 - \left(\frac{B}{2}\right)^2$$

* use to solve (QE)s & derive (QF)

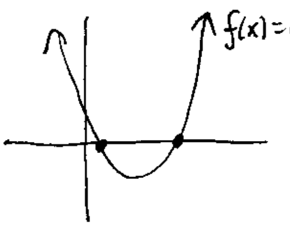
Connection between QE & functions

QE: $ax^2 + bx + c = 0$ ← an equation (at most 2 solutions)

Quadratic Function:

$f(x) = ax^2 + bx + c$ ← a function! input x, output $f(x)$

Graph of Quadratic Function is parabola opening up/down:

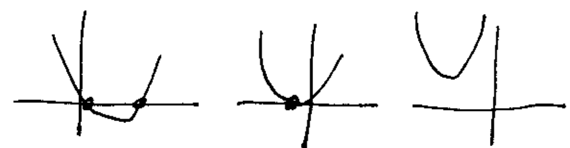


CONNECTION: zeros of the function are the Solutions of the equation!

$ax^2 + bx + c = 0$

Discriminant: $\Delta = b^2 - 4ac$

- $\Delta > 0 \rightsquigarrow$ QE has 2 real solutions
- $\Delta = 0 \rightsquigarrow$ QE has 1 real solution only
- $\Delta < 0 \rightsquigarrow$ QE has no real solutions (but has



Cube Roots Given $a \in \mathbb{R}$.

Solutions to $x^3 = a$ are $x = a^{1/3} = \sqrt[3]{a}$.

Note: a can be negative!

Radicals

Given $a \in \mathbb{R}$, $n \in \mathbb{N}$ the n^{th} root of a, denoted $\sqrt[n]{a}$ or $a^{1/n}$, is the real solution to the equation: $x^n - a = 0$, if it exists.

Or it can defined by: it is the real number whose nth power is a (ie $(\sqrt[n]{a})^n = a$)

Note nth roots don't always exist!

- $a \geq 0, n > 0$: $x^n - a = 0$ has only one non-negative solution denoted $\sqrt[n]{a}$ or $a^{1/n}$.
- $a \geq 0, n > 0$, n odd: $x^n - a = 0$ has only one solution.
- $a > 0, n > 0$, n even: $x^n - a = 0$ has two solutions: $\sqrt[n]{a}$ and $-(\sqrt[n]{a})$.
- $a < 0$, n odd: $x^n - a = 0$ has one solution & $\sqrt[n]{a}$ is negative
- $a < 0$, n even: $x^n - a = 0$ has $\boxed{\text{no}}$ real solutions so $\sqrt[n]{a}$ is undefined!

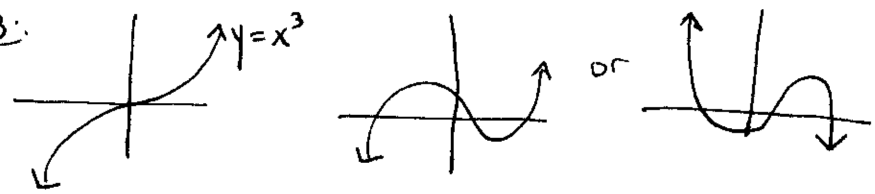
Polynomials

- $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, $\quad a_n, a_{n-1}, \ldots, a_2, a_1, a_0 \in \mathbb{Q}$ called coefficients
- n – highest power of x – called degree of polynomial
- defined for all real values of x

Degree 1: $p(x) = a_1 x + a_0$ linear function (Graph line)

Degree 2: $p(x) = a_2 x^2 + a_1 x + a_0$ Quadratic Function (Graph parabola up/down)

Degree 3:



Degree 4:

Rational Functions

- $r(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ & $q(x)$ are two polynomials
- defined only for values of x such that $q(x) \neq 0$.
- Graphs can be complicated! Study in Ch3!

Roots roots of a polynomial are when $p(x) = 0$, ie zeros of $p(x)$.
roots of a rational function when $r(x) = 0$, but only need to solve for numerator $= 0$

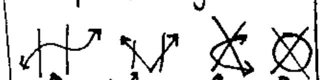
FUNCTIONS

(1) FORMAL DEF.
a function f is a set of ordered pairs (x, y) any two ordered pairs with same x-values must have the y-value.

So

(2) Since we can represent (x, y) as points on the plane, graphically a function can ~~only have one point on~~ intersect a vertical line only once! This condition is called VLT — vertical Line Test

FOUR WAYS

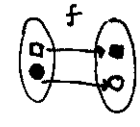
(1) formal def ordered pairs with only one x-value allowed	(2) Graphs passing VLT  YES! NO
(3) EQuations in 2 variables where for each x, only one solution for y Eg. $4x + y = 6$ $x^2 = 2y$ $x^5 + 3y = 0$	(4) FUNCTION NOTATION x = input $f(x)$ = output $f: X \to Y$ $x \mapsto f(x)$ Eg $f(x) = 6 - 4x$ $f(x) = \frac{1}{2}x^2$ $f(x) = \frac{\lvert x-4 \rvert}{x-4}$, $x \neq 4$

DOMAIN: set of x-values either implicitly or explicitly given
D(f)

RANGE: set of y-values, ie outputs, corresponding to the domain
R(f)

(3) If using equations to define a function, VLT can be phrased as: for each fixed x-value there's only one y-solution.
[$x^2 = 2y \to y = x^2/2$ ✓ (pass)
$x^2 = 2y^2 \to y = \pm\sqrt{x^2/2}$ ✗ (fails)]

(4) input x $\longrightarrow$ output $f(x)$



Notation $f: \underbrace{X}_{\text{DOMAIN}} \longrightarrow \underbrace{Y}_{\text{RANGE}}$

CONNECTION

$(x, y) \in f \Leftrightarrow y = f(x) \Leftrightarrow (x, f(x)) \in f$

$x \in D(f)$

$y \in R(f)$

Domains: implied domain → means all allowed values of x. Eg $f(x)=\frac{1}{x^2-1}$ implied domain is $D(f)=\{x\in\mathbb{R} \mid x\neq 1,-1\}$

explicit domain → means it's restricted by choice

Eg $f(x)=\frac{1}{x^2-1}$, $x>5$.

so $D(f)=\{x\in\mathbb{R} \mid x>5\}$.

Piece-wise Functions

- useful way to represent different pieces of different formulas but of course must be careful that VLT is passed.

Eg $f(x)=\begin{cases} 1-x, & x<1 \\ \sqrt{x-1}, & x\geq 1 \end{cases}$

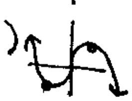
Note: two expressions don't overlap! $x<1$ & $x\geq 1$ no intersection

Eg $f(x)=\begin{cases} 1+x, & x\leq 1 \\ \sqrt{x-1}, & x\geq 1 \end{cases}$

NOT A FUNCTION!

- Since $1+1=2$ & $2\neq 0$
  $\sqrt{1-1}=0$
- problem $x=1$: the two formulas $1+x$ & $\sqrt{x-1}$ don't agree here!

EVEN $f(-x)=f(x)$ (ie symmetry across y-axis) 

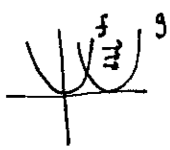
ODD $f(-x)=-f(x)$ (ie symmetry across origin) 

TRANSFORMATIONS

- Vertical Shifts: 

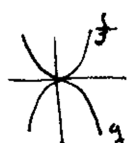
  - original: $f(x)$
  - new: $g(x)=f(x)+a$

  (a constant) $a>0$ shift up
  $a<0$ shift down

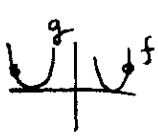
- Horizontal Shifts: 

  - original: $f(x)$
  - new: $g(x)=f(x-a)$
    - $a>0$ shift right
    - $a<0$ shift left

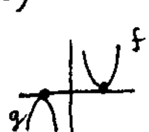
Reflection

- Across x-axis: 

  - original: $f(x)$
  - new: $g(x)=-f(x)$

- Across y-axis: 

  - original: $f(x)$
  - new: $g(x)=f(-x)$

- Across origin: 

  - original: $f(x)$
  - new: $g(x)=-f(-x)$

COMBINING FUNCTIONS

Given two functions f & g: can form new functions

- $(f+g)(x)=f(x)+g(x)$ Domain $=D(f)\cap D(g)$
- $(f-g)(x)=f(x)-g(x)$
- $(f\cdot g)(x)=f(x)\cdot g(x)$
- $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$, $g(x)\neq 0$.
- FUNCTION COMPOSITION: $f\circ g$ & $g\circ f$

$$\boxed{(f\circ g)(x)=f(g(x))} \quad \& \quad \boxed{(g\circ f)(x)=g(f(x))}$$

Best described by the input/output schematics:

$$\begin{tikzcd} \underset{\text{input}}{x} \arrow[r, "g"] \arrow[dr, dashed, "f\circ g"'] & \underset{\text{output/input}}{g(x)} \arrow[d, "f"] \\ & f(g(x)) \end{tikzcd} \qquad \begin{tikzcd} \underset{\text{input}}{x} \arrow[r, "f"] \arrow[dr, dashed, "g\circ f"'] & \underset{\text{output/input}}{f(x)} \arrow[d, "g"] \\ & g(f(x)) \end{tikzcd}$$

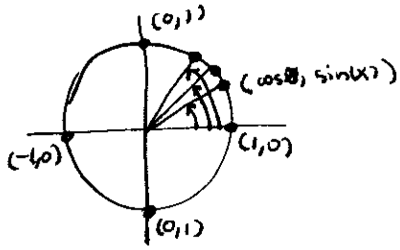
- composition "chains" together two functions
- MUST BE CAREFULL WITH DOMAIN!

Intercepts

x-intercepts: when $f(x)=0$ ie the roots or zeros of f

y-intercept: $y=f(0)$ only one y-intercept

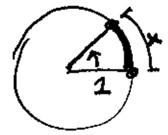
TRIGONOMETRY



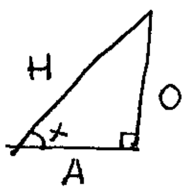
UNIT CIRCLE

- x = measure of length of the arc on unit circle from $(1,0)$ to the point $(\cos(x), \sin(x))$.

denoted like an angle but units are <u>radians</u> not degrees!



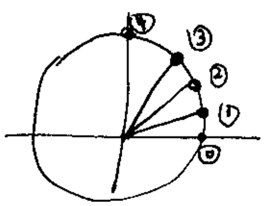
- DEGREE TO RADIAN CONVERSION: $360^\circ = 2\pi$, or $180^\circ = \pi$ radians, or $\frac{\pi \text{ radians}}{180 \text{ degrees}} = 1$ (unit-less)



- For right triangle w/ angle $0 < x < \pi/2$, define

$\boxed{\sin(x) = \frac{O}{H}}$ $\boxed{\cos(x) = \frac{A}{H}}$ $\boxed{\tan(x) = \frac{O}{A}}$.

SOH CAH TOA



- SPECIAL VALUES

① $30^\circ = \frac{\pi}{6}$ radians: $\left(\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \ \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}\right)$

② $45^\circ = \frac{\pi}{4}$ radians: $\left(\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \ \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\right)$

③ $60^\circ = \frac{\pi}{3}$ radians: $\left(\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, \ \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}\right)$

⓪ $0^\circ = 0$ radians: $\left(\cos(0) = 1, \ \sin(0) = 0\right)$

④ $90^\circ = \frac{\pi}{2}$ radians: $\left(\cos\left(\frac{\pi}{2}\right) = 0, \ \sin\left(\frac{\pi}{2}\right) = 1\right)$

OTHER TRIG FUNCTIONS:

$\boxed{\tan(x) = \frac{\sin(x)}{\cos(x)}}$ $\boxed{\cot(x) = \frac{\cos(x)}{\sin(x)}}$

$\boxed{\sec(x) = \frac{1}{\cos(x)}}$

$\boxed{\csc(x) = \frac{1}{\sin(x)}}$

INEQUALITIES:

$\boxed{-1 \leq \sin(x) \leq 1}$

$\boxed{-1 \leq \cos(x) \leq 1}$

IDENTITIES:

$\boxed{\sin^2(x) + \cos^2(x) = 1}$

essentially same:

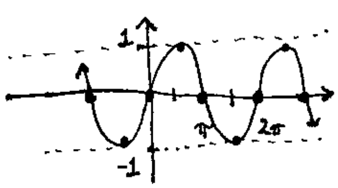
$1 + \tan^2(x) = \sec^2(x)$

$1 + \cot^2(x) = \csc^2(x)$

SINE ODD: $\sin(-x) = -\sin(x)$

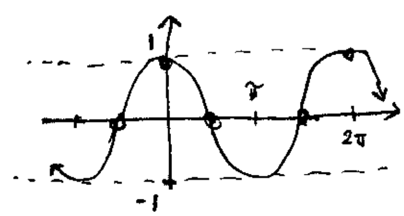
COSINE EVEN: $\cos(-x) = \cos(x)$

GRAPHS



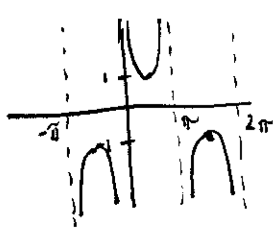
$y = \sin(x)$

Domain: $\mathbb{R}$



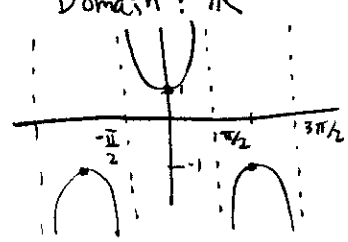
$y = \cos(x)$

Domain: $\mathbb{R}$

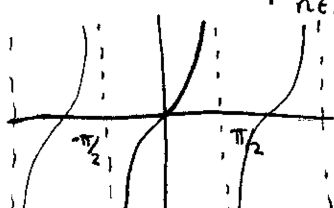


$y = \csc(x)$

Domain: $\{x \in \mathbb{R} \mid x \neq \pi n, n \in \mathbb{Z}\}$

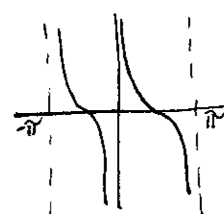


Domain: $\{x \in \mathbb{R} \mid x \neq \frac{\pi}{2} + \pi n, n \in \mathbb{Z}\}$



$y = \tan(x)$

Domain: same as $\sec(x)$, ie when $\cos(x) \neq 0$



$y = \cot(x)$