PITZER COLLEGE

MATH 30: Calculus I — SPRING 2018

Notes — Dr. Basilio

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This document contains:

- Exam day info
- FINAL Exam day info
- Exam 1, Exam 2 and Exam 3 topics covered list
- Chapter 1, 2, 3, 4 notes
- Exam 1, Exam 2 and Exam 3 Practice Problems
- Exam 1, Exam 2 and Exam 3 Practice Problems Answers to even problems

What to expect on Exam day

- I'll arrive to our classroom before 9:00 am and we'll have a Q&A where I'll answer any questions you have until 9:30 am. Then you'll bring all of your belongings to the front of the classroom and take Exam 1 from 9:30-10:30am.
- So, the length of time is 60 minutes. Though I'll usually allow an extra 5-10 minutes if you want/need the time.
- You cannot use any calculator or electronic devise during the exam.
- Once the exam starts you may not use the restroom. So please use the restroom before the exam starts or during the first 30 minutes.
- Expect a mix of True/False, Multiple Choice, and Free Response questions.

What to expect for Final Exam

- Monday, May 7 from 9 am - 12pm in our usual classroom.
- I'll arrive to our classroom before 8:30 am and we'll have a Q&A where I'll answer any questions you have until 8:55 am. Then you'll bring all of your belongings to the front of the classroom and take the Final Exam from 9 am - 12 pm.
- You cannot use any calculator or electronic devise during the exam.
- Students will get one bathroom break, must turn in their exam while they leave the room, and only one student at a time.

- Expect a mix of True/False, Multiple Choice, and Free Response questions. See the Practice Problems below.
- It is cumulative exam, so everything we covered is fair game. We'll have a slight focus on anti-differentiation and integration 4.9 and Chapters 5 material. I'd estimate 40% will cover material from 4.9 & Ch 5 and 60% will cover material from Ch 1-4.9 (differentiation).

Material Covered

EXAM 1: Monday, February 12

Chapter 1: Functions and Models

1.1 - Four Ways to Represent a Function
1.2 - Mathematical Models: A Catalogue of Essential Functions
1.3 - New Functions from Old Functions
1.4 - Exponential Functions
1.5 - Inverse Functions and Logarithms

Chapter 2: Limits and Derivatives

2.1 - The Tangent and Velocity Problems
2.2 - The Limit of a Function
2.3 - Calculating Limits using the Limit Laws
2.4 - The Precise Definition of a Limit*
2.5 - Continuity
2.6 - Limits at Infinity; Horizontal Asymptotes
2.7 - Derivatives and Rates of Change
2.8 - The Derivative as a Function

EXAM 2: March 23

Exam 1 Material

Chapter 3: Differentiation Rules

3.1 - Derivatives of Polynomials and Exponential Functions
3.2 - The Product and Quotient Rules
3.3 - Derivatives of Trigonometric Functions
3.4 - The Chain Rule
3.5 - Implicit Differentiation
3.6 - Derivatives of Logarithmic Functions
3.7 - Rates of Change in the Natural and Social Sciences
3.8 - Exponential Growth and Decay
3.9 - Related Rates

EXAM 3: April 27

Chapter 3: Differentiation Rules

3.10 - Linear Approximations and Differentials

Chapter 4: Applications of Differentiation

4.1 - Maximum and Minimum Values
4.2 - The Mean Value Theorem
4.3 - How Derivatives Affect the Shape of a Graph
4.5 - Summary of Curve Sketching
4.7 - Optimization Problems
4.9 - Antiderivatives

Chapter 5: Integrals

5.1 - Areas and Distances

FINAL EXAM: May 7

Exam 1, 2, & 3 Material

Chapter 4: Applications of Differentiation

4.9 - Antiderivatives

Chapter 5: Integrals

5.1 - Areas and Distances
5.2 - The Definite Integral
5.3 - The fundamental Theorem of Calculus
5.4 - Indefinite Integrals and the Net Change Theorem

Notes

Chapter 1: Functions and Models

- Refer to the Precalculus Review notes for more details
- Know the fundamental concepts: FUNCTION, domain, range, independent variable, dependent variable, vertical line test
- Be comfortable using INTERVAL NOTATION to describe the domain of a function
- Four ways to represents a function:
  1) definition/using words 2) using a table/graphs 3) equations 4) function notation
- piece-wise defined functions, absolute value function (know this definition)
- increasing, decreasing functions
- types of functions: constant functions, linear functions, polynomial functions (know terms: degree, roots), power functions, radical/root functions, rational functions, algebraic functions, trigonometric functions, exponential functions, logarithmic functions
- be familiar with examples of each of the above functions and know how to generate their graphs
- TRANSFORMATION of functions:
  vertical translations, horizontal translations, vertical dilations/stretching, horizontal dilations/stretching, reflection across the x-axis, reflection across the y-axis
- COMBINATION of functions:
  know what the following are and how to find their domains: $(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x)$, $(f/g)(x)$, $(f \circ g)(x)$, $(g \circ f)(x)$
- EXPONENTIAL functions: $f(x) = b^x$, base $b > 0$, $b \neq 1$.
  Know their properties, graphs according to two cases: Case I: $b > 1$ & Case II: $0 < b < 1$
- be able to work with the "Laws of Exponents" on page 47
- Example 3 on page 50-51 on Half-Lifes
- THE NUMBER e – READ HAND-OUT

- INVERSE functions: they exists if and only if they are one-to-one if and only if they pass the horizontal line test; how to find algebraically and graphically; inverse properties
- domain of f^{-1} = range of f; range of f^{-1} = domain of f
- LOGARITHMIC functions: $f^{-1}(x) = \log_b(x)$ are the inverse functions to the exponential functions $f(x) = b^x$
  Know their properties, graphs according to two cases: Case I: $b > 1$ & Case II: $0 < b < 1$
- Log Eq $\leftrightarrow$ Exp Eq
- Know: Laws of Logarithms on page 60
- Natural logarithm = $\ln(x) = \log_e(x)$.
- Know the graph of the SIX TRIGONOMETRIC FUNCTIONS: $\sin(x)$, $\cos(x)$, $\tan(x)$, $\cot(x)$, $\sec(x)$, $\csc(x)$
- INVERSE TRIGONOMTERIC functions:
  $\sin^{-1}(x) = \arcsin(x)$, $\cos^{-1}(x) = \arccos(x)$, $\tan^{-1}(x) = \arctan(x)$ and know their domain and ranges and graphs:

- $$\sin^{-1}x = y \iff \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

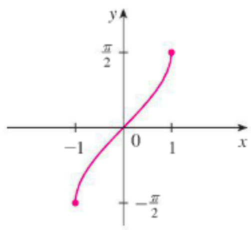


FIGURE 20
$y = \sin^{-1}x = \arcsin x$

- $$\cos^{-1}x = y \iff \cos y = x \quad \text{and} \quad 0 \leqslant y \leqslant \pi$$

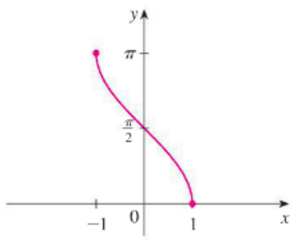
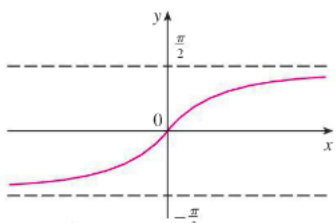


FIGURE 22
$y = \cos^{-1}x = \arccos x$

- $$\tan^{-1}x = y \iff \tan y = x \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$



$y = \tan^{-1}x = \arctan x$

- **11**
  $$y = \csc^{-1}x \ (|x| \geqslant 1) \iff \csc y = x \quad \text{and} \quad y \in (0, \pi/2] \cup (\pi, 3\pi/2]$$
  $$y = \sec^{-1}x \ (|x| \geqslant 1) \iff \sec y = x \quad \text{and} \quad y \in [0, \pi/2) \cup [\pi, 3\pi/2)$$
  $$y = \cot^{-1}x \ (x \in \mathbb{R}) \iff \cot y = x \quad \text{and} \quad y \in (0, \pi)$$

Chapter 2: Limits and Derivatives

• Section 2.1: The Tangent and Velocity Problems

TANGENT LINE PROBLEM (TLP): Given a curve $y = f(x)$ and a point $P = (a, f(a))$ on the curve, find the "line that best fits the curve near the point P." This is called the TANGENT LINE of the curve $y = f(x)$ at the point P.

- Know what a secant line is. If $y = f(x)$ is a function and $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are two points on the curve, then the secant line is the straight line passing through these two points. The slope is written m_{sec} or m_{PQ}
- To find the equation of the tangent line T you only need to find the slope, $m_{\tan}(P)$, because you already know the x and y coordinates of the point P and you can use the point-slope formula to find the equation of the line.
- We solve the TLP using the infinite PROCESS to find the slope $m_{\tan}(P)$: (1) find an approximate solution with a nearby point Q close to P, (1 1/2) find a better approximate solution using another point Q that is closer to P than in the previous step, (2) limit process:
$$\boxed{m_{\tan}(P) = \lim_{Q \to P} m_{\text{sec}} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}}$$

INSTENTANEOUS VELOCITY PROBLEM (IVP): Given a moving object that is traveling a distance of $s(t)$ after t seconds, find the velocity of the object at the instant t_1.

- Know what average velocity is. If $y = s(t)$ is a function that tells us the distance traveled and t_1 and t_2 are two different times, then the interval of time is $[t_1, t_2]$. The change in time is written $\Delta t = t_2 - t_1$. The distance traveled *between* t_1 *and* t_2 is $\Delta s = s(t_2) - s(t_1)$. Thus the AVERAGE VELOCITY over the interval $[t_1, t_2]$ is:
$$\boxed{v_{avg} = \frac{\text{distance}}{\text{time}} = \frac{\Delta s}{\Delta t}}$$
- The instantaneous velocity of the object at the instant t_1 is denoted by $v_{inst}(t_1)$
- We solve the IVP using the infinite PROCESS to find the instantaneous velocity $v_{inst}(t_1)$: (1) find an approximate solution with a nearby time t_2 close to t_1, (1 1/2) find a better approximate solution using another time t_2 that is closer to t_1 than in the previous step, (2) limit process:
$$\boxed{v_{inst}(t_1) = \lim_{t_2 \to t_1} v_{avg} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}}$$

Note the similarity between the solutions to these two problems!

• Section 2.2: The Limit of a Function

- DEFINITION OF A LIMIT: $\boxed{\lim_{x \to a} f(x) = L}$ means "as x approaches a, but $x \neq a$, either from smaller or larger values, the values of $f(x)$ approach the single value L"
This definition guarantees we avoid the problems with functions that are not defined at a, ie $f(a)$ is undefined, and replaces the KEY IDEA with the INFINITE PROCESS of approximation. The values of x are approaching a but never allowed to equal a and the limit L is the TREND that $f(x)$ seems to be approaching.
When the above fails, we say the limit "does not exist" and put $\boxed{\lim_{x \to a} f(x) = DNE}$

- One-Sided Limits:
  $\boxed{\text{LHL: } \lim_{x\to a^-} f(x) = L}$ key is $\boxed{x < a}$ as x approaches a
  $\boxed{\text{RHL: } \lim_{x\to a^+} f(x) = R}$ key is $\boxed{x > a}$ as x approaches a

- For example, if

$$f(x) = \begin{cases} -2x+6, & \text{if } x < 1 \\ 3x+1, & \text{if } x \geq 1 \end{cases}$$

  Then the left-handed limit as $x \to 1$ is $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(-2x+6) = -2(1)+6 = 4$. Why did we choose $f(x) = -2x+6$? because as $x \to 1^-$, only values of x strictly less than 1 are considered so $x < 1$. The right-handed limit is $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(3x+1) = 3(1)+1 = 4$ because $x > 1$. Notice that LHL $\neq$ RHL.

- EXISTENCE of a limit: $\lim_{x\to a} f(x) = L$ exists if and only if LHL=RHL
  (ie. if $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$)

- INFINITE LIMITS: $\boxed{\lim_{x\to a} f(x) = +\infty}$ means "as x approaches a, but $x \neq a$, either from smaller or larger values, the values of $f(x)$ becomes arbitrarily large positive"
  Other ways to express this: "$f(x)$ becomes infinite positive as x approaches a" or "$f(x)$ increases without bound as x approaches a"

- INFINITE LIMITS: $\boxed{\lim_{x\to a} f(x) = -\infty}$ means "as x approaches a, but $x \neq a$, either from smaller or larger values, the values of $f(x)$ becomes arbitrarily large negative"
  Other ways to express this: "$f(x)$ becomes infinite negative as x approaches a" or "$f(x)$ decreases without bound as x approaches a"

- The main examples are:

$$\lim_{x\to a+}\left(\frac{1}{x-a}\right) = \frac{1}{+0} = +\infty \quad \lim_{x\to a-}\left(\frac{1}{x-a}\right) = \frac{1}{-0} = -\infty$$

  and, more generally,

$$\lim_{x\to a+}\left(\frac{1}{(x-a)^n}\right) = \frac{1}{+0} = +\infty$$

$$\lim_{x\to a-}\left(\frac{1}{(x-a)^n}\right) = \begin{cases} \frac{1}{+0} = +\infty, & \text{if } n \text{ is even} \\ \frac{1}{-0} = -\infty, & \text{if } n \text{ is odd} \end{cases}$$

- SHORTCUTS: $\boxed{\frac{\pm C}{\pm 0} = \pm\infty}$ This is really four different possibilities: (here $C \neq 0$ is constant)

$$\boxed{\frac{+C}{+0} = +\infty} \quad \boxed{\frac{+C}{-0} = -\infty} \quad \boxed{\frac{-C}{+0} = -\infty} \quad \boxed{\frac{-C}{-0} = +\infty}$$

- VERTICAL ASYMPTOTES (VAs): A function can have many vertical asymptotes. Vertical asymptotes are vertical lines, so they have the form $x = a$. By definition, the line $x = a$ is a vertical asymptote if any of the following occurs: $\lim_{x\to a-} f(x) = \pm\infty$ or $\lim_{x\to a+} f(x) = \pm\infty$. Intuitively $f(x)$ tries to HUG the vertical lines $x = a$.
  How to find VAs: (1) look for values a where denominator=0, (2) check that num$\neq 0$ at a
  (this makes sense in light of the above shortcuts for finding infinite limits)

- HOW CAN LIMITS FAIL TO EXIST: $\boxed{\lim_{x\to a} f(x) = L}$
  1. LHL $\neq$ RHL
  2. Wild Oscillation
  3. Infinite

- CAREFUL: infinite limits are technically DNE! Though we write $\lim_{x\to a} f(x) = \pm\infty$ to indicate the trend (it grows or decreases without bound), it doesn't approach a real number L. Thus, we consider infinite limits as DNE.

- Section 2.3: Calculating Limits Using the Limit Laws

  - Know the statement of the eleven LIMIT LAWS on pages 95, 96, 97:
    Suppose that c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exists.

LIMIT LAWS
1. $\lim_{x\to a}[f(x) + g(x)] = [\lim_{x\to a} f(x)] + [\lim_{x\to a} g(x)]$
2. $\lim_{x\to a}[f(x) - g(x)] = [\lim_{x\to a} f(x)] - [\lim_{x\to a} g(x)]$
3. $\lim_{x\to a}[cf(x)] = c[\lim_{x\to a} f(x)]$
4. $\lim_{x\to a}[f(x) \cdot g(x)] = [\lim_{x\to a} f(x)] \cdot [\lim_{x\to a} g(x)]$
5. $\lim_{x\to a}[\frac{f(x)}{g(x)}] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ (if $\lim_{x\to a} g(x) \neq 0$)
6. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$
7. $\lim_{x\to a}[c] = c$
8. $\lim_{x\to a}[x] = a$
9. $\lim_{x\to a}[x^n] = a^n$ (n a positive integer)
10. $\lim_{x\to a}[\sqrt[n]{x}] = \sqrt[n]{a}$ (if $\sqrt[n]{a}$ exists)
10. $\lim_{x\to a}[\sqrt[n]{f(x)}] = \sqrt[n]{\lim_{x\to a} f(x)}$ (if exists)

  - DIRECT SUBSTITUTION: whenever $\lim_{x\to a} f(x) = f(a)$ ie "you can just plug-in $x = a$ into $f(x)$" "DSub" works for lots of functions! Like polynomials and rational functions as long as a is in the domain

  - LIMITS AND $\frac{0}{0}$: when DSub fails and you get $\frac{0}{0}$ the limit can still exist and take on any value. There's two methods to know: (1) Cancellation method; (2) Conjugate method

    Example of Cancellation method: $\lim\limits_{h\to 0} \dfrac{(3+h)^2 - 9}{h} = \frac{0}{0}$

$$\begin{aligned}
\lim_{h\to 0} \frac{(3+h)^2 - 9}{h} &= \lim_{h\to 0} \frac{(9 + 6h + h^2) - 9}{h} \\
&= \lim_{h\to 0} \frac{6h + h^2}{h} \\
&= \lim_{h\to 0} 6 + h \qquad \text{(cancel the } h \text{ is ok bc } h \neq 0\text{)} \\
&= 6 \qquad \text{(evaluate the limit using LLs)}
\end{aligned}$$

Example of Conjugate method: $\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2} = \frac{0}{0}$

$$\begin{aligned}
\lim_{h\to 0}\frac{\sqrt{t^2+9}-3}{t^2} &= \lim_{h\to 0}\frac{\sqrt{t^2+9}-3}{t^2}\cdot\frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3} && \text{(multiply by conjugate)}\\
&= \lim_{h\to 0}\frac{(\sqrt{t^2+9}-3)(\sqrt{t^2+9}+3)}{t^2(\sqrt{t^2+9}+3)}\\
&= \lim_{h\to 0}\frac{(t^2+9)-9}{t^2(\sqrt{t^2+9}+3)} && \text{(foil top, pro tip: leave bottom un-multiplied)}\\
&= \lim_{h\to 0}\frac{t^2}{t^2(\sqrt{t^2+9}+3)} && \text{(notice we can cancel the } t^2\text{)}\\
&= \lim_{h\to 0}\frac{1}{\sqrt{t^2+9}+3} && \text{(cancel the } t^2 \text{ is ok bc } t\neq 0\text{)}\\
&= \frac{1}{\sqrt{0+9}+3} = \frac{1}{6} && \text{(evaluate the limit using LLs)}
\end{aligned}$$

- Know: Examples 7 and 8 on page 100
- SQUEEZE Theorem: if $f(x)\leq g(x)\leq h(x)$ for all values close to a except possibly at a itself AND $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$, THEN $\lim_{x\to a} g(x) = L$
- Be able to use the Squeeze Theorem to show that $\lim_{x\to 0} x^2\sin(\frac{1}{x}) = 0$ (example 11)

- Section 2.5: Continuity

- Memorize the definition of Continuity: The function $f(x)$ is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$. This means that the following THREE conditions are ALL satisfied:

  (I) $f(a)$ is defiined
  (II) $\lim_{x\to a} f(x)$ exists (i.e. LHL=RHL)
  (III) $f(a) = \lim_{x\to a} f(x)$

- Be comfortable, with the alternative definition of continuity as well (replace II-III with II') $\lim_{h\to 0} f(a+h)$ exists, III') $f(a) = \lim_{h\to 0} f(a+h)$).
- To check if a function is continuous at a point $x = a$, you must verify that (I), (II), and (III) are ALL true. However, to show that a functions is NOT continuous at a point $x = a$, AT LEAST ONE of the conditions (I), (II), or (III) needs to FAIL.
- Points of DISCONTINUITY. We say that $x = a$ is a removable discontinuity of the function f if $\lim_{x\to a} f(x)$ exists (ie LHL=RHL), but $f(a)\neq \lim_{x\to a}$. We say that $x = a$ is a non-removable discontinuity) of the function f if $\lim_{x\to a} f(x)$ does NOT exist (ie LHL≠RHL). In other words, you see a "gap" between the function near $x = a$.
- Continuous on an interval= f is continuous at every point inside the interval
- One-sided continuity:
  LHC: f is continuous at $x = a$ from the left if $\lim_{x\to a^-} f(x) = a$.
  RHC: f is continuous at $x = a$ from the right if $\lim_{x\to a^+} f(x) = a$.

- Continuity Theorem(s):
  (T5a) Polynomials are continuous everywhere, ie on interval $(-\infty, +\infty)$.
  (T5b) Rational functions are continuous on their domains (open intervals).
  (T7) Root functions, Trigonometric functions, Inverse Trigonometric functions, Exponential functions, Logarithmic functions are ALL continuous on their domains (open intervals).
  (T8/9) The composition of two continuous functions is continuous on its domain For example, $P(x) = x^4 - 15x + 6$ is continuous at all real numbers. (Why? Check that all 3 conditions are true!)

- Intermediate Value Theorem (IVT):
  Assume that f is continuous on the closed interval $[a, b]$ and $f(a) \neq f(b)$. Then f assumes every values between $f(a)$ and $f(b)$. More precisely, if N is any number such that $f(a) < N < f(b)$, then there is some number $c \in (a, b)$ (i.e. $a < c < b$) so that $f(c) = N$.

- Know how to show that a polynomial equation, or more complicated equation, has a root by using the IVT as in Example 10 on page 123.

- Section 2.6: Limits at Infinity; Horizontal Asymptotes

- LIMITS AT INFINITY: $\boxed{\lim_{x\to\pm\infty} f(x) = L}$ means "as x grows arbitrarily large, the values of $f(x)$ approach the single value L"
  This means that the graph of the function $y = f(x)$ HUGS the horizontal like $y = L$ for large positive values of x.
  We also say "as x approaches positive infinity" to express $x \to +\infty$ even though x does not approach any real number.

- LIMITS AT INFINITY: $\boxed{\lim_{x\to\pm-\infty} f(x) = L}$ means "as x decreases arbitrarily negatively large, the values of $f(x)$ approach the single value L"
  This means that the graph of the function $y = f(x)$ HUGS the horizontal like $y = L$ for large negative values of x.
  We also say "as x approaches negative infinity" to express $x \to -\infty$ even though x does not approach any real number.

- SHORT CUTS: $\boxed{\frac{\pm C}{\pm\infty} = 0}$ This is really four different possibilities: (here $C \neq 0$ is constant)

  $\boxed{\frac{+C}{+\infty} = 0}$ $\boxed{\frac{1}{-\infty} = 0}$ $\boxed{\frac{1}{+0} = +\infty}$ $\boxed{\frac{1}{-0} = -\infty}$

  NOTE WELL: These are not really equal signs, but limits in disguise.

- Know what a DOMINANT TERM of a rational function is.

- The main theorem for computing limits at infinity is
  Dominant Term Theoream (DTT) Whenever you take the limit of a rational function AT INFINITY, you can simply look at the dominant terms:

$$\lim_{x\to\pm\infty} \left(\frac{\boxed{a_n x^n} + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0}{\boxed{b_k x^k} + b_{k-1}x^{k-1} + \cdots + b_1 x^1 + b_0} \right) = \lim_{x\to\pm\infty} \left(\frac{a_n x^n}{b_k x^k} \right) = \begin{cases} \frac{a_n}{b_n}, & \text{if } n = k \\ 0, & \text{if } n < k \\ \pm\infty, & \text{if } n > k \end{cases}$$

  In the last case, when $n > k$, you need to be careful and pay attention to signs. Use the shortcuts for finding infinite limits from section 2.2.

- HORIZONTAL ASYMPTOTES (HAs): A function can have at most two horizontal asymptotes. Horizontal asymptotes are horizontal lines, so they have the form $y = c$. By definition, the line $y = R$ is a *horizontal asymptote (from the right)* if $\lim_{x\to+\infty} f(x) = R$ and
the line $y = L$ is a *horizontal asymptote (from the left)* if $\lim_{x\to+\infty} f(x) = L$. Intuitively $f(x)$ tries to HUG the horizontal line $y = L$ or $y = R$ for very large x values.
How to find: compute $\lim_{x\to+\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$

- INFINITE LIMITS AT INFINITY: this is when $\boxed{\lim_{x\to\pm\infty} f(x) = \pm\infty}$
Many functions do this like polynomials.

- Study Examples 4 and 5 on pages 130-131

- Section 2.7: Derivatives and Rates of Change

  - TLP: we solve the tangent line problem at $P = (a, f(a))$ by finding the slope of the tangent line:

$$m_{\tan}(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h}$$

because as x approaches a, the point $Q = (x, f(x))$ on the curve $y = f(x)$ approaches the point P so the above limit is really the same thing as: $m_{\tan}(P) = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}$ from section 2.1

  - EQUATION OF THE TANGENT LINE:
$\boxed{y - f(a) = m_{\tan}(a)) \cdot (x - a)}$ or $\boxed{y = f(a) + m_{\tan}(a)) \cdot (x - a)}$

  - IVP: we solve the instantaneous velocity problem at any time $t = a$ by:

$$v_{inst}(a) = \lim_{t\to a} \frac{s(t) - s(a)}{t - a} = \lim_{h\to 0} \frac{s(a + h) - s(a)}{h}$$

because as t approaches a, the average velocity of the object over the interval of time $[a, t]$ is $v_{avg} = \frac{s(t)-s(a)}{t-a}$, so the above limit is really the same thing as: $v_{ins}(a) = \lim_{\Delta t\to 0} \frac{\Delta s}{\Delta t}$ from section 2.1

  - Both Ancient Problems are solved in the same way, we give this a new name "derivative"

  - The DERIVATIVE OF A FUNCTION AT $x = a$:

$$f'(a) = \left.\frac{df}{dx}\right|_{x=a} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h}$$

So, $\boxed{m_{\tan}(a) = f'(a)}$ and $\boxed{v_{ins}(a) = f'(a)}$

  - INSTENTANEOUS RATE OF CHANGE (IROC): IROC= limit(AROC)
average rate of change of a function is $AROC = \frac{f(x_2)-f(x_1)}{x_2-x_1}$ so

$$IROC = \lim_{\Delta x} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \lim_{\Delta x\to 0} \frac{\Delta f}{\Delta x} = \left.\frac{df}{dx}\right|_{x=x_1}$$

This is the same as the derivative so why another way to write it? Example 6 on page 146 is why

- Section 2.7: Derivatives and Rates of Change

  - DERIVATIVE AS A FUNCTION: $f'(x) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$
The domain of the derivative function is the set of all values a for which $f'(a)$ exists. In this case, we say that f is DIFFERENTIABLE at $x = a$.

- Be able to sketch the values of $f'(x)$ quickly from a graph and determine the graph of $f'(x)$.
- Be able to find $f'(x)$ using the limit definition. We studied limits in great detail so that we can find derivatives! Now's your chance to use that theory :-)
- THEOREM (DIFFERENTIABLE IMPLIES CONTINUOUS): If f is differentiable at $x = a$, then f is continuous at $x = a$.
  This is a very useful theorem, because computing a derivative is much easier than having to very all 3 conditions in the definition of continuity. You should still check condition (I), but then if the derivative exists at this point, the function will be continuous there. Once we learn how to compute derivatives quickly using shortcuts this theorem becomes more powerful!
- **WARNING:** The converse to this theorem is NOT true. This means that you may have a function that is continuous at $x = a$ but does not have a derivative there. The easiest example of this is the absolute value function at $x = 0$.
- HOW CAN $f'(a)$ DNE?
  1. f not continuous at $x = a$
  2. f has a vertical tangent line at $x = a$, i.e. $\lim_{x\to a} |f'(a)| = \pm\infty$
  3. f has a cusp/corner at $x = a$, i.e. LHL $\neq$ RHL
- HIGHER DERIVATIVES: $f''(x) = \frac{d^2 f}{dx^2}$, $f'''(x) = \frac{d^3 f}{dx^3}$, $\ldots$, $f^{(n)}(x) = \frac{d^n f}{dx^n}$
- ACCELERATION: $a(t) = v'(t) = \frac{ds}{dt}$

Chapter 3: Differentiation Rules

- Section 3.1: Derivatives of Polynomials and Exponential Functions

- f is a **differentiable function** whenever we know that the corresponding derivative function $f'(x)$ exists for some interval of values.
- COMMON NOTATIONS FOR DERIVATIVES:

$$f'(x) = m_{\tan}(x) = v_{ins}(x) = \frac{df}{dx}$$

  Here's are few more that you should know:

$$f'(x) = y' = \frac{dy}{dx} = \dot{s} \text{ and } f'(x) = \frac{d}{dx}\left[\ f(x)\ \right] = D_x[f(x)] = Df(x)$$

- Derivative Rules: Part 1

  DR1: Constant Rule: $\frac{d}{dx}[C] = 0$
  DR2: Line Rule: $\frac{d}{dx}[ax+b] = a$
  DR3: Power Rule: $\frac{d}{dx}[x^n] = nx^{n-1}$
  DR4: Constant Multiplier Rule: $\frac{d}{dx}[Cf(x)] = Cf'(x)$
  DR5: Sum/Difference Rule: $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
  DR: Exp Rule: $\frac{d}{dx}[e^x] = e^x$
- Using DRs 1-5, we can find the derivative of any polynomial function
- A polynomial $p(x)$ of degree n, can have at most $(n-1)$ turning points because it can have at most $(n-1)$ HTLs, ie. solutions to $\frac{dy}{dx} = 0$.

- SVA: position=$s(t)$, velocity=$v(t) = s'(t)$, acceleration=$a(t) = v'(t) = s''(t)$
- Definition of e: the slope of the tangent line of $f(x) = e^x$ at $(0, 1)$ is 1
- The derivative of e^x is itself!

- Section 3.2: The Product and Quotient Rules
  - Derivative Rules: Part 2

    DR6: Product Rule: $\dfrac{d}{dx}[f(x) \cdot g(x)] = f'(x)g(x) + f(x)g'(x)$

    DR7: Quotient Rule: $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$
  - Know how to use these DRs individually and also in combination with the previous DRs
  - HIGHER DERIVATIVES: $f''(x) = \frac{d^2 f}{dx^2}$, $f'''(x) = \frac{d^3 f}{dx^3}$, $\ldots$, $f^{(n)}(x) = \frac{d^n f}{dx^n}$

- Section 3.3: Derivatives of Trigonometric Functions
  - Derivative Rules: Part 3

    DR8: Sine Rule: $\dfrac{d}{dx}[\sin(x)] = \cos(x)$ DR9: Cosine Rule: $\dfrac{d}{dx}[\cos(x)] = -\sin(x)$
  - Know how find very high number of derivatives of sine or cosine functions, e.g $\dfrac{d^{27}}{dx^{27}}[\cos(x)]$
  - Special Trig limits:
$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \qquad \text{and} \qquad \lim_{x \to 0} \frac{\cos(x) - 1}{x} = 0$$
  - Know how to use the special trig limits to evaluate limits
  - Know how to prove the Sine Rule (DR8) and the Cosine Rule (DR9) using the trig identities and the special trig limits. The trig identities will be provided so you don't have to memorize them but you do for the special trig limits.
  - DR10: Trig Rule:

    $\dfrac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$ $\dfrac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$

    $\dfrac{d}{dx}[\tan(x)] = \sec^2(x)$ $\dfrac{d}{dx}[\cot(x)] = -\csc^2(x)$
  - Know how to derive the formulas from DR10 using the Quotient Rule and the Sine and Cosine DRs

- Section 3.4: The Chain Rule
  - Review composite functions, be able to recognize the "inside" and "outside" functions
  - DR11: Chain Rule: $\dfrac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$
  - Be able to compute derivatives using the chain rule and previous rules in combination
  - DR12: General Exp Rule: $\dfrac{d}{dx}[b^x] = b^x \cdot \ln(b)$

- Section 3.5: Implicit Differentiation

  - Understand the difference between an implicitly defined function and an explicitly defined function
  - Know how to use the technique of implicit differentiation to find the derivative of a function defined implicitly
  - Be able to use implicit diff to find all HTLs and VTLs
  - DR 13: Inverse Trig Rule:

$$\frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left[\cos^{-1}(x)\right] = \frac{-1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\left[\tan^{-1}(x)\right] = \frac{1}{1+x^2} \qquad \frac{d}{dx}\left[\cot^{-1}(x)\right] = \frac{-1}{1+x^2}$$
$$\frac{d}{dx}\left[\csc^{-1}(x)\right] = \frac{-1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\left[\sec^{-1}(x)\right] = \frac{1}{x\sqrt{x^2-1}}$$

- Section 3.6: Derivatives of Logarithmic Functions

  - DR14: Natural Log Rule: $\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}$
  - Know the proof of DR 14
  - DR15: Log Rule: $\frac{d}{dx}\left[\log_b(x)\right] = \frac{1}{x}\cdot\frac{1}{\ln(b)}$
  - Know the proof of DR 15
  - Recall that logarithms have very useful properties:
    (1) $\log_b(A \cdot B) = \log_b(A) + \log_b(B)$
    (2) $\log_b\left(\frac{A}{B}\right) = \log_b(A) - \log_b(B)$
    (3) $\log_b(A^r) = r \cdot \log_b(A)$
  - Be able to use the log properties to simplify the computation of derivatives of functions
  - Know how to use the technique: Logarithmic Differentiation:
    Step 1: Write the function as $y = f(x)$.
    Step 2: Apply $\ln[\quad]$ to both sides of the equation from Step 1.
    Step 3: Simplify using the log properties as much as possible.
    Step 4: Implicitly differentiate both sides
    Step 5: Solve for $\frac{dy}{dx}$.

- Section 3.7: Rates of Change in the Natural and Social Sciences

  - The change in x by $\Delta x = x_2 - x_1$
    The change in our function f by $\Delta f = f(x_2) - f(x_1)$
  - The average rate of change of f over the interval $[x_1, x_2]$ is given by

$$AROC = \frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

- The derivative is the limit of the average rate of change as the interval shrinks to zero, or as $x_2 \to x_1$:

$$\left.\frac{df}{dx}\right|_{x_1} = \lim_{x_2 \to x_1} \frac{\Delta f}{\Delta x}.$$

- if x_2 is close to x_1 then the average rate of change is approximately the derivative at x_1, so:

$$\left.\frac{df}{dx}\right|_{x_1} \approx \frac{\Delta f}{\Delta x} \qquad \text{(for } x_2 \approx x_1\text{)}$$

- Sign of the derivative tells you the function is increasing/decreasing:
  if the derivative is positive at $x = x_1$, then the numerator of the AROC is positive for x_2 close to x_1 so $f(x_1) < f(x_2)$. Thus f is increasing for points close to x_1!
  Similarly, if the derivative is negative at $x = x_1$, then for x_2 close to x_1 the function f is decreasing for points close to x_1!

- Follow that particle!
  Know the following: velocity, acceleration, particle at rest, particle moving forward/backwards, diagram to represent motion, distance traveled, speeding up/down

- Something's fishy:
  Stable population

- Economics:
  The derivative is called the **marginal cost** and since

$$C'(x) = \frac{dC}{dx} \approx C(x+1) - C(x)$$

  the marginal cost of producing x items is approximately equal to the cost of producing one more unit–the $(x+1)$th unit!
  The units of marginal cost are units of currency per units of items

- Supply/Demand

- Smooth Operator:
  position ($s(t)$), velocity ($v(t) = s'(t)$), acceleration ($a(t) = v'(t) = s''(t)$), Jerk $J(t) = a'(t) = s'''(t)$), snap ($S(t) = J'(t) = s''''(t)$).

- Section 3.8: Exponential Growth and Decay

  - Recall exponential functions $f(t) = b^t$. When $b > 1$ we call this exponential growth, when $0 < b < 1$ we call this exponential decay (See Chapter 1 for a review)

  - Theorem: The solutions to the "natural growth/decay equation," $\frac{df}{dt} = kf(t)$,
    are of the form: $f(t) = Ce^{kt}$. The constant $C = f(0)$ is called the initial condition (or amount, or population, etc).

  - Connection of exponential functions, e^{kt}, with exponential functions base b, b^t:
    Since $e^{kt} = (e^k)^t$ we have that $b = e^k$ so that when $k > 0$, $b > 1$ (look at the graph of e^x to convince yourself of this), and when $k < 0$, then $e^k = \frac{1}{e^{|k|}}$ so that $b = \frac{1}{e^{|k|}} < 1$.
    This is why we also call $f(t) = e^{kt}$ exponential growth for $k > 0$ and exponential decay for $k < 0$.

  - Differential Equations: an equation involving a function and its derivatives. The goal in solving a differential equation is to find the function or functions that make the equation true when substituted.

- Know how to set-up equations for doubling time and half-life
- Predator and Preys: just be able to understand how an equation with derivatives can be interpreted (p. 9 on worksheet)
- Compounding Interest:
  Interest is compounded n times a year: $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$
  Interest is compounded continuously: $A(t) = Pe^{rt}$
  Keep in mind, the interest rate r needs to be in decimal form in formulas

● Section 3.9: Related Rates

- How to solve "Related Rates Problems (RRPs)":
  Step 1: Read the problem to determine key terms and units.
  Step 2: Determine:
  ● All quantities that are changing in time (also which are constant in time).
  ● Assign appropriate mathematical notation to these quantities.
  ● Draw a diagram if possible
  ● Translate all of the information given in the problem into mathematical notation, called the "Givens".
  ⋆ Use the units to help you match it with the correct quantities.
  ⋆ Use the key term "rate" as code for derivative.
  Step 3: Find an equation or equations that relates the quantities from Step 1.
  Step 4: Implicitly Differentiate both sides of the main equation from Step 2 with $\frac{d}{dt}$
  Step 5: Substitute the "givens" into the resulting equation from Step 3 and solve for the unknown rate.
- The "EME System" highlights all of the essential steps in the process of solving word problems and emphasizes the importance of contextualizing your final answers in writing:
  Step 1: "English" (E)
  Step 2: "English to Math" $(E \to M)$
  Step 3: "Math" (M)
  Step 4: "Math to English" $(M \to E)$

● Section 3.10: Linearization and Differentials

- A function f is **locally linear near at** $x = a$ whenever f is differentiable at $x = a$. That is, $f'(a)$ exists.
- The local linearization of f at $x = a$ is the linear function: $\boxed{L(x) = f(a) + f'(a)(x - a)}$
- Linear Approximation of f at $x = a$: $\boxed{\text{If } x \approx a, \text{ then } f(x) \approx L(x)}$
- Know how to apply the above linearization to approximate functions as in the examples from the worksheets.
- The **differential of a variable** x, dx, is another independent variable. It is equivalent to think of dx as just any real number. However, it is usually used as the change in x: $\boxed{dx = \Delta x = x_2 - x_1.}$
- If $y = f(x)$ and f is a differentiable function, then we defined the **differential of** y, dy, by the equation: $\boxed{dy = f'(x)dx.}$ Sometimes I will also use the notation: $df = f'(x)dx$.

- Meaning of differential: The differential $dy = f'(x)dx$ approximates the actual change in y, Δy, if x changes by dx.

- Know how to compute differentials and also use them to approximate the change in a function.

Chapter 4: Applications of Differentiation

● Section 4.1: Maximum and Minimum Values

- Local vs Global Extrema:
  ● Given a function f, we say that $f(c)$ is a **global (or absolute) maximum value of** f provided that $\boxed{f(c) \geq f(x)}$ for all x in the domain of f. Similarly, we call $f(c)$ a **global (or absolute) minimum value of** f provided that $\boxed{f(c) \leq f(x)}$ for all x in the domain of f.
  ● Given a function f, we say that $f(c)$ is a **local (or relative) maximum value of** f provided that $\boxed{f(c) \geq f(x)}$ for all x near c. More precisely, there exists a small interval I containing c where $f(c)$ is the global maximum on I. Similarly, we call $f(c)$ a **local (or relative) minimum value of** f provided that $\boxed{f(c) \leq f(x)}$ for all x near c.
  ● Any maximum or minimum may be called an **extreme value of** f.
  ● By **extrema** of f we mean either the maximum or minimum value of f.

- Critical Numbers/Values/Points:
  ● We call $x = c$ a **critical number** of a function f if either $f'(c) = 0$ or $f'(c) =$ DNE.

  ● If c is a critical number of f, we call $(c, f(c))$ a **critical point** of f and $f(c)$ a **critical value** of f.

- Fermat's Theorem: IF f has a local extrema at $x = c$ THEN c is a critical number of f.

- Extreme Value Theorem (EVT): A continuous function on a closed interval must attain both its maximum and minimum values at some points inside the interval.

  More precisely, if f is continuous on the closed interval $[a, b]$, then there exists number $c \in [a, b]$ where $f(c)$ is the global maximum value of f on $[a, b]$ and there exists a number $d \in [a, b]$ where $f(d)$ is the global minimum value of f on $[a, b]$.

- How to find extrema on a closed interval:
  (This is called the "closed interval method" in the book).

  Let f be a continuous function on the closed interval $[a, b]$. To find the extrema:

  Step 1: Find all the critical numbers $c \in (a, b)$ and evaluate f at these numbers.
  Step 2: Evaluate f at the endpoints, i.e. find $f(a)$ and $f(b)$.
  Step 3: Select the largest and smallest values from the list of values found in Steps 1 and 2.

● Section 4.2: Rolle's and Mean Value Theorem

- Rolle's Theorem:
  Assume that we have a function f on the closed interval $[a, b]$ that satisfies the following assumptions:

  (1) f is continuous on $[a, b]$

  (2) f is differentiable on (a, b)

  (3) $f(a) = f(b)$

  THEN there exists a number $c \in (a, b)$ where $f'(c) = 0$.

- Know the statement of Rolle's Theorem and be able to verify if a function satisfies the conditions of the theorem

- Know how to solve Activity 2, that it to show that a polynomial has at most one root.

- Mean Value Theorem:
  Assume that we have a function f on the closed interval $[a, b]$ that satisfies the following assumptions:
  (1) f is continuous on $[a, b]$
  (2) f is differentiable on (a, b)
  THEN there exists a number $c \in (a, b)$ such that $\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$
  or, equivalently, $\boxed{f(b) - f(a) = f'(c)(b - a).}$

- Know the statement of Mean Value Theorem and be able to verify if a function satisfies the conditions of the theorem

- Section 4.3: Maximum and Minimum Values

- Theorem: Derivative zero implies constant:
  • IF $f'(x) = 0$ for every x in an interval (a, b), THEN f is constant on (a, b).
  • IF $f'(x) = g'(x)$ for every x in an interval (a, b), THEN $f(x) = g(x) + C$ on (a, b) for a constant C.

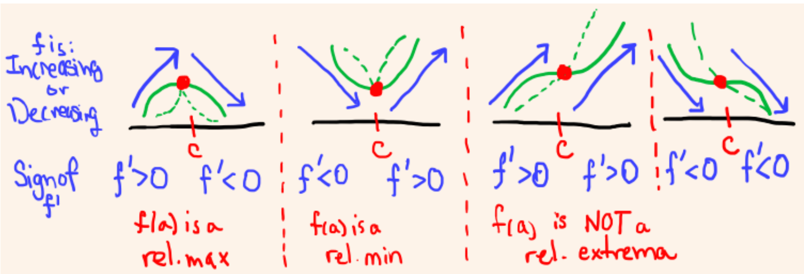
- The increasing/decreasing Test (ID Test):
  • If $f'(x) > 0$ for every value of x inside an open interval I, THEN f is increasing on the interval I.
  • If $f'(x) < 0$ for every value of x inside an open interval I, THEN f is decreasing on the interval I.

- Be able to use the ID Test to determine where a function is increasing/decreasing

- First Derivative Test (FDT):
  Let f be a continuous function and c a critical number of f.
  • IF f' changes from positive to negative at c, THEN f has a local maximum at c.
  • IF f' changes from negative to positive at c, THEN f has a local minimum at c.
  • IF f' is positive to the left and to the right of c, THEN f does NOT have a relative extrema at c.
  • IF f' is negative to the left and to the right of c, THEN f does NOT have a relative extrema at c.

- I call the FDT: "the local/relative extrema hunter"

- CSI Lines for f':



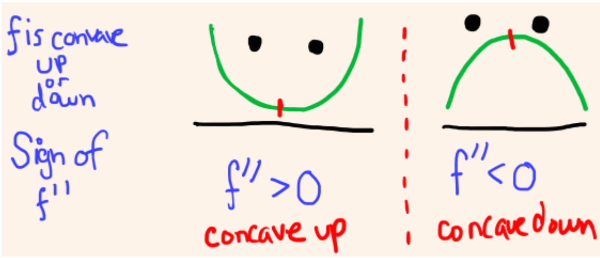
- By building a CSI Line for f' you can apply the First Derivative Test and find LOCAL/RELATIVE extrema.

- Be sure to be able to build CSI lines for f' and to be able to sketch a function given a complete CSI Line for f'.

- Concavity and Points of Inflection:
  • A function f is **concave UP** on an open interval I if the graph of f lies ABOVE all of its tangent lines on the interval I.
  Geometrically this means: the curve is bending up!
  • A function f is **concave DOWN** on an open interval I if the graph of f lies BELOW all of its tangent lines on the interval I.
  Geometrically this means: the curve is bending down!
  • A point $P = (a, f(a))$ is a **point of inflection** if the function f changes concavity at P either from positive to negative, or negative to positive.

- Concavity Test:
  • IF $f''(x) > 0$ for every value of x inside an open interval I, THEN f is concave UP on the interval I.
  • IF $f''(x) < 0$ for every value of x inside an open interval I, THEN f is concave DOWN on the interval I.

- CSI Lines for f'':



- Second Derivative Test (FDT):
  Suppose that f'' exists and is continuous near c.
  • IF $f'(c) = 0$ and $f''(c) < 0$, THEN f has a local maximum at c.
  • IF $f'(c) = 0$ and $f''(c) > 0$, THEN f has a local minimum at c.
  • IF $f'(c) = 0$ and $f''(c) = 0$, THEN the test fails and anything can happen.
  In this case, use the First Derivative Test.

• Section 4.5: Summary of Curve Sketching

- CSI List:
  When sketching a function f by hand. Consider:

  Tools from precalculus:

  (1) Domain/Continuity

  (2) Intercepts (x and y)

  (3) Symmetry (if any), Periodicity (if any)

  Tools from calculus:

  (1) Asymptotes (HAs, VAs)

  (2) CSI Line for f': ID Test/First Derivative Test, Local Extrema

  (3) CSI Line for f'': Cocavity Test, Points of Inflection

- Be able to sketch graphs of functions by using the CSI List. On some questions, you might be given a filled out CSI List and be asked to sketch a function from that info.

- Section 4.7: Optimization Problems

- An **optimization problem (OP)** is a problem where one seeks the best solution, either the minimum or maximum value to a mathematical model (or where they occur).
  A **Primary Equation (PE)** associated to an OP is an equation involving the quantity that is to be optimized.
  A **Constraint (or Secondary) Equation (SE)** associated to an OP is an equation(s) that constrains the quantities involved to be within certain bounds.

- How to solve "Optimization Problems (OPs)":
  Step 1: Read the problem to determine key terms and units.

  Step 2: Draw a diagram and introduce notation.

  Step 3: Determine the Primary Equation (PE).
  - Identify the quantity which is to be optimized.

  Step 4: Determine the Constraint/Secondary Equation (C/SE).
  - Identify the constraints, if any.
    - ⋆ The constraints could be intervals, or equations.
  - Another purpose of the constraint equation is that it helps reduce the problem to a one-variable problem.

  Step 5: Solve the OP by finding the extrema of the quantity that is to be optimized from Step 3. Don't forget units!
  - Pro tip: Usually solving for $f'(x) = 0$ is enough in applications since it is obvious which CN will be the extrema sought. In other words, it is usually unnecessary to fill out complete CSI Lines and use the FDT or SDTs.
  - Pro tip: If you determined an appropriate interval in Step 4, don't forget to check the end-points!

- Economic Optimization Problems: Cost, Revenue, Profit:
  Key: Marginal functions are derivatives
  The **revenue function** is simply: $R(x) = px$, where p is the price of the items sold. The units of p are *currency per items*, so that the revenue function has the units of currency.
  In simple problems the price is constant but there are also situations where it is more complicated. When the price is a function of the number of items demanded, we write this as $p(x)$. Economists call this a **demand function**. Thus, we expect that as x (number of items sold) increases, the price p decreases. It makes more sense to think of it as: when the price decreases, the number of items sold increases. The **marginal revenue** is the derivative of the revenue, $R'(x)$.
  The **profit function** is the revenue minus the costs: $P(x) = R(x) - C(x)$.

- This section is super important and will be an emphasis of Exam 3. Be sure to study all the examples given in the worksheets and be able to set-up the problems correctly. I will ask problems where all I want are the primary and secondary equations (i.e. the set-up) but you will not have to solve the entire problem. This way I can test on many more types of problems.

- Section 4.9: Anti-derivatives

  - Anti-derivatives:
    Given a function $f(x)$ defined on an interval I, **an anti-derivative of** f is a function $F(x)$ (if it exists) defined also on I whose derivative is $f(x)$ for all x in I. That is, we seek to find $F(x)$ such that
    $$F'(x) = f(x), \qquad x \in I.$$
    The process of **anti-differentiation** is the inverse of the process of differentiation.

  - Key Insight: Just like subtraction is the inverse operation of addition and division is the inverse operation of multiplication, anti-differentiation is the inverse operation of differentiation.

  - Slope Fields:
    A geometric interpretation of anti-derivatives is that of **slope fields**. Since $F'(x) = f(x)$ is given, we know what the slope of F is at every point in the interval I.

  - Theorem: All Anti-derivatives:
    IF $F(x)$ is an anti-derivative of $f(x)$, THEN all other anti-derivatives of $f(x)$ are of the form: $F(x)+C$.

  - Guess and Check method for finding anti-derivatives

  - Anti-derivatives Notation:
    Given a function $f(x)$ defined on an interval I, we denote all the anti-derivatives by $F(x) + C$ or by using the symbols $\int f(x)dx$. Thus,
    $$\int f(x)dx = F(x) + C$$
    Anti-derivatives are also called **indefinite integrals**.

  - Theorem: Derivatives and Anti-Derivatives:
    The definition of an anti-derivative immediately implies that
    $$\frac{d}{dx}\int f(x)\,dx = f(x)$$
    and
    $$\int \frac{d}{dx}[f(x)]\,dx = f(x) + C$$

  - Theorem: A Few Important Anti-Derivative Rules
    - ADR 3: Anti-Power Rule: If $n \neq -1$, then $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$.
    - ADR14: Natural Log Rule: $\int \frac{1}{x}\,dx = \ln(x) + C$.
    - ADR 4: Anti-Constant Multiplier Rule: $\int kf(x)\,dx = k\int f(x)\,dx$.
    - ADR 5: Anti-Sum/Diff Rule: $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$.
    - ADR: Anti-Exp Rule: $\int e^x\,dx = e^x + C$

- ADR 8: Anti-Sine Rule: $\int \sin(x)\,dx = -\cos(x) + C$
- ADR 9: Anti-Cosine Rule: $\int \cos(x)\,dx = \sin(x) + C$

- Be able to find anti-derivatives using the ADRs.
- Solving ODEs:
  Given an ODE of the form, $\frac{dy}{dx} = f(x)$, where $f(x)$ is a given function, the **general solutions** are $y(x) = \int f(x)\,dx$.

  $$\frac{dy}{dx} = f(x) \quad \Longrightarrow \quad y(x) = \int f(x)\,dx.$$

  Notice the general solution is actually infinitely many solutions because of the $+C$.
  A **particular solution**, is a solution to the ODE that passes through a specific point (x_0, y_0). The given (x_0, y_0) are also called **initial conditions**.
- Galileo's Discovery:
  Terrestrial objects undergoing **free-falling** motion (no external forces) have the same constant acceleration.
  Using units the acceleration is measured to be $a(t) = -32$ feet per second squared or $a(t) = -9.8$ meters per second squared for any object undergoing free-falling motion.
- Newton's Equations of Motion:
  If a terrestrial object undergoes free-falling motion, then its equations of motion are:

  $$\begin{aligned} a(t) &= -32 & a(t) &= -9.8 \\ v(t) &= -32t + v_0 & v(t) &= -9.8t + v_0 \\ s(t) &= -16t^2 + v_0 t + s_0, & s(t) &= -4.9t^2 + v_0 t + s_0, \end{aligned}$$

  where $v_0 = v(0)$ is the initial velocity and $s_0 = s(0)$ is the initial position.

  They are the same equations just in different units.
- Be able to solve equations of motion (SVA) given the acceleration by anti-differentiating (twice).

Chapter 5: Integration

- Section 5.1: Area and Distance Problems

- Know the definition of area some basic formulas from geometry
- Area Problem using Sums of Rectangles:
  Let n be a positive integer, $\boxed{\Delta x = \frac{b-a}{n}}$, sub-intervals $[x_{i-1}, x_i]$, $\boxed{x_i = a + i\Delta x}$ for $i = 0, 1, 2, \ldots, n$.

  If we select a point $x_i^* \in [x_{i-1}, x_i]$, that is, $x_{i-1} \le x_i^* \le x_i$, the we can view these as a "random sample."

  When f is non-negative on the interval $[a, b]$, the **area under a curve** $y = f(x)$ **bounded by the lines** $x = a$, $x = b$, **and the** x**-axis** is given by

$$\boxed{A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \left(f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x\right)}$$

• When x_i^* is the right-endpoint then $x_i^* = x_i$. We say it is a **right-endpoint approximation to the area**.
• When x_i^* is the left-endpoint then $x_i^* = x_{i-1}$. We say it is a **right-endpoint approximation to the area**.
• When x_i^* is the midpoint of each subinterval then $x_i^* = (x_{i-1} + x_i)/2$. We say it is a **midpoint approximation to the area**.
• When the x_i^* are chosen so that $f(x_i^*)$ is the minimum value of f on the subintervals, then we say that A_n is a **lower sum, or lower-estimate**.
• When the x_i^* are chosen so that $f(x_i^*)$ is the maximum value of f on the subintervals, then we say that A_n is a **upper sum, or upper-estimate**.

- Two important observations:
  • When f is INCreasing on the interval (a, b), then the left-endpoint approximations for A_n are upper-estimates for the exact area A, and the right-endpoint approximations for A_n are lower-estimates for the exact area A.

  • When f is DECreasing on the interval (a, b), then the left-endpoint approximations for A_n are upper-estimates for the exact area A, and the right-endpoint approximations for A_n are lower-estimates for the exact area A.

- Distance Problem using Sums of *velocity* $\times$ *time*:
  Let n be a positive integer, $\boxed{\Delta t = \dfrac{b-a}{n}}$, sub-intervals $[t_{i-1}, t_i]$, $\boxed{t_i = a + i\Delta t}$ for $i = 0, 1, 2, \ldots, n$.

  If we select a point $t_i^* \in [t_{i-1}, t_i]$, that is, $t_{i-1} \le t_i^* \le t_i$, the we can view these as a "random sample."
  When $v(t)$ is non-negative on the interval $[a, b]$, the **total distance traveled by a particle with velocity** $v(t)$ **from** $t = a$ **to** $t = b$ is given by

$$\boxed{D = \lim_{n\to\infty} D_n = \lim_{n\to\infty} \left(v(t_1^*)\Delta t + v(t_2^*)\Delta t + \cdots + v(t_n^*)\Delta t\right)}$$

- The area under the graph of the velocity function between the two line $t = a$ and $t = b$ is equal to the distance traveled by an object from $t = a$ to $t = b$.

• Section 5.32 The Definite Integral

- Sigma Notation for summation: $\displaystyle\sum_{i}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$.

- Memorize and Know how to use: A Few Basic Summation Formulas
  • $\displaystyle\sum_{i=1}^{n}[ca_i] = c\sum_{i=1}^{n} a_i$, for a constant c
  • $\displaystyle\sum_{i=1}^{n}[a_i \pm b_i] = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$
  • $\displaystyle\sum_{i=j}^{n} a_i = a_j + a_{j+1} + \cdots + a_n$, provided that $i \le j \le n$.

- Know how to use the following formulas:

  • Sum of a constant: $\displaystyle\sum_{i=1}^{n} c = c \cdot n$, for a constant c

  • Sum of the first n integers: $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

  • Sum of the square of the first n integers: $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

  • Sum of the cube of the first n integers: $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

  NOTE: YOU DO NOT NEED TO MEMORIZE THESE – they will be provided on the Final Exam.

- DEFINITE INTEGRALS:
  Let f be a function defined on the closed interval $[a, b]$.

  Let n be a positive integer, $\boxed{\Delta x = \frac{b-a}{n}}$, sub-intervals $[x_{i-1}, x_i]$, $\boxed{x_i = a + i\Delta x}$ for $i = 0, 1, 2, \ldots, n$.

  If we select a point $x_i^* \in [x_{i-1}, x_i]$, that is, $x_{i-1} \leq x_i^* \leq x_i$, the we can view these as a "random sample."

  The **definite integral of f from $x = a$ to $x = b$** is defined to be

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x}$$

  provided this limit exists and gives the same value for all possible choices of sample points.

  • When the definite integral exists, we say f is **integrable** on $[a, b]$.

  • When the definite integral Does Not Exist, we say f is **non-integrable** on $[a, b]$.

  • We call f the **integrand**, and a and b the **limits of integration** with a called the **lower limit** and b called the **upper limit**.

  • The process of finding the definite integral is called **integration**.

  • The expression $\displaystyle\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called a **Riemann Sum**.

- IMPORTANT THEOREMS: Area and Distance Problems and Definite Integrals
  • When $f(x)$ is non-negative over $[a, b]$, then the area under the graph of f bounded by the lines $x = a$, $x = b$, and the x-axis is given by:

$$\text{Area} = \int_a^b f(x)\,dx.$$

- When $v(t)$ is non-negative over $[a, b]$, then the distance traveled by the object from $t = a$ to $t = b$ is given by:

$$\text{Distance} = \int_a^b v(t)\,dt.$$

- Understand what the definite integral is when f is non-negative. Called **net (or signed) area**.
- THEOREM: Continuous functions are integrable

  If f is continuous on $[a, b]$, then f is integrable. That is, the definite integral, $\int_a^b f(x)\,dx$, exists and is equal to a finite value.
- THEOREM: Properties of the Definite Integral
  - $\int_a^a f(x)\,dx = 0.$
  - $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$, when $a < b$.
  - $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, when $a < c < b$.
- Fundamental Theorem of Calculus

PRACTICE EXAMS

PRACTICE PROBLEMS FOR EXAM 1

Chapter 1 Review Problems: pages 68-70

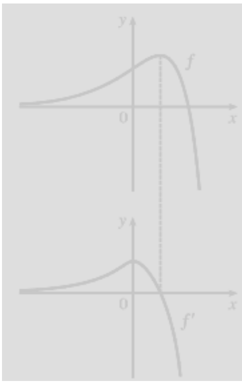
- Concept Check: 1-13 all
- True-False Quiz: 1-14 all
- Exercises: 1, 3, 5-8, 11, 14, 19, 20, 23, 25, 26, 27

Chapter 2 Review Problems: pages 165-168

- Concept Check: 1-16 all
- True-False Quiz: 1-26 all
- Exercises: 1–5, 7, 9,13,15,17, 19, 23, 24, 29, 33, 35, 40, 41-43, 45a,b, 47, 48

Answers: Ch 1 R
CC will be uploaded separately (see Sakai)
T/F: FFFTT FFTTF FFFF
Ex For odd answers see back of book.
Even: 6. $D = [-2, 2]$ 8. $D = (-\infty, \infty)$ 14. Shift $y = \ln(x)$ left 1 unit. 20. Let $h(x) = x + \sqrt{x}$, $g(x) = \sqrt{x}$, $f(x) = 1/x$. Then $F(x) = (f \circ g \circ h)(x)$. 26. (a) $x = \ln(5)$ (b) $x = e^2$ (c) $x = \ln(\ln(2))$ (d) $x = \tan(1)$
Answers: Ch 2 R
CC will be uploaded separately (see Sakai)
T/F: FFTFT TFFTF TFTFF FTTTF FTFTT F
Ex For odd answers see back of book.
Even: 4. 0 24. Let $f(x) = -x^2$, $g(x) = x^2\cos(1/x^2)$, $h(x) = x^2$. Because $-1 \leq \cos(x) \leq 1$ for all x, we have $-1 \leq \cos(1/x^2) \leq 1$ for all $x \neq 0$. Multiplying by x^2 and noting that $x^2 > 0$ for $x \neq 0$ we have $-x^2 \leq x^2\cos(1/x^2) \leq x^2$ for all $x \neq 0$. Thus, $f(x) \leq g(x) \leq h(x)$ for $x \neq 0$. Because $\lim_{x\to 0}(-x^2) = 0$ and $\lim_{x\to 0}(x^2) = 0$, by the Squeeze Theorem, we conclude that $lim_{x\to 0}g(x) = 0$. Thus, $\lim_{x\to 0} x^2\cos(1/x^2) = 0$. 40. $f(x) = x^6$ and $a = 2$. 48. The tangent line of a has positive slope for $x < 0$ and negative slope for $x > 0$ so this matches with the values of graph b. Notice that at $x = 0$, the tangent line of graph a is horizontal so it's slope is zero and the graph of b is zero at 0. Thus $a = f$ and $b = f'$. The graph of b has horizontal tangent lines to the left and the right of the y-axis and at these values the graph of c is zero so that c is the derivative curve for b. Since $b = f'$, $c = f''$.

2.

42. 

PRACTICE PROBLEMS FOR EXAM 2

Chapter 3 Review Problems: pages 266-269

- Concept Check: 1-6 all
- True-False Quiz: 1-15 all
- Exercises: 1-10, 13-15,17,21,25,28,30,41, 51, 52, 53 just y' only, 56-59, 60, 61 just tangent line, **65**, 66, 69, 70, 71, 75, 77, 83, 85, 89, 92, 93, 94, 95, 97, 99

Answers: Ch 3 R
CC will be uploaded separately (see Sakai)

T/F: TFTTF FFFTF TTTFT
Ex For odd answers see back of book.
Even: 2. $y' = -\frac{1}{2x^{3/2}} + \frac{3}{5x^{8/5}}$ or $y' = -\frac{1}{2x\sqrt{x}} + \frac{3}{5x\sqrt[5]{x^3}}$. 4. $\frac{(1+\cos(x))\sec^2(x)+\tan(x)\sin(x)}{(1+\cos(x))^2}$.
6. $y' = \cos^{-1}(x) - \frac{x}{\sqrt{1-x^2}}$. 8. $\frac{dy}{dx} = \frac{y\cos(x)-e^y}{xe^y-\sin(x)}$. 10. $\frac{dy}{dx} = e^{mx}(m\cos(nx) - n\sin(nx))$. 14. $\frac{dy}{dx} = \tan(x)$.
28. $\frac{dy}{dx} = (\cos(x))^x(\ln(\cos(x)) - x\tan(x))$. 30. Use log diff! $\frac{dy}{dx} = (\frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5})(\frac{8x}{x^2+1} - \frac{6}{2x+1} - \frac{15}{3x-1})$.
52. $g''(\pi/6) = \sqrt{3} - \pi/12$. 56. Use special trig limits. Ans: 1/8. 58. $y = -1$. 60. $y - 1 = -4/5(x-2)$.
70. (a) $P'(2) = -2$ (b) $Q'(2) = -3/8$ (c) $C'(2) = 6$. 92. (a) $C'(x) = 2 - 0.04x + 0.00021x^2$ (b) $C'(100) = 2 - 4 + 2.1 = -0.1$ \$/unit, This value represents the rate at which costs are increasing as the hundredth unit is produced, and is the approximate cost of producing the 101st unit. (c) The cost of producing the 101st item is $C(101) - C(100) = 990.10107 - 990 = \$0.10107$, slightly larger than $C'(100)$.
94. (a) $m(t) = Ce^{kt}$, $k = \ln(1/2)/5.24$, $m(20) = 100e^{k20} \approx 7.1$ mg (b) $1 = 100e^{kt}$, $t = \ln(1/100)/k = 5.24(\ln(1/2)/\ln(1/100))$.

PRACTICE PROBLEMS FOR EXAM 3

REMARK: The book review problems are not a very good match for what I have in mind for Exam 3. So it is best to study directly from the worksheets and class notes to get a better idea of the problems I'll ask. In particular, for the optimization problems, the online homework and worksheets will be the best preparation.

Chapter 3 Review Problems: pages 266-269

- Concept Check: 7
- True-False Quiz:
- Exercises: 103(a), 104

Answers: Ch 3 R
CC will be uploaded separately (see Sakai)
Ex For odd answers see back of book.
Even: 104. $y = x^3 - 2x^2 + 1$ then $dy = (3x^2 - 4x)dx$. When $x = 2$ and $dx = 0.2$, $dy = [3(2)^2 - 4(2)](0.2) = 0.8$.

Chapter 4 Review Problems: pages 358-359

- Concept Check: 1-6, 9, 11
- True-False Quiz: 1-9
- Exercises: 1,5, 15, 19, 21, 23, 27, 45, 59, 65, 67, 69, 71, 73,

Answers: Ch 4 R
CC will be uploaded separately (see Sakai)
Ex For odd answers see back of book.
Even:

PRACTICE PROBLEMS FOR FINAL EXAM

Chapter 5 Review Problems: pages 421–424

- Concept Check: 1, 2, 4, 7,8
- True-False Quiz: 1, 6, 11, 12, 13, 14, 15, 17
- Exercises: 1a, 2 all parts, 5, 7, 8, 9, 11, 13, 15, 45, 58,

Answers: Ch 5 R
CC will be uploaded separately (see Sakai)
Ex For odd answers see back of book.
Even: 2. 8. 58.