SARAH • LAWRENCE • COLLEGE

HHW# 1 - Midterm #1 FALL 2016

Math 3005. Calculus 1. Jorge Eduardo Basilio

Posted on Wednesday Morning.

DIRECTIONS

There are THREE Parts to this assignment, weighted according to:

Part 1 - 20 points
Part 2 - 120 points
Part 3 - 60

• Part 1: True or False?

Please solve all TWENTY True/False questions in Part 1. Each is worth 1 point.

• Part 2: ABCs

This is my take on a "choose your own adventure" exam. There are 10 problems. Each problem itself has three versions of a similar problem, labeled (A), (B), or (C) and they correspond to an "A-level," "B-level," or "C-level" understanding of the Calculus (chapters 1 and 2). (A) problems are worth 12 points, (B) problems are worth 8 points, and (C) problems are worth 4 points.

For each problem, pick ONE AND ONLY ONE PART A, B, or C to solve. Therefore, you should do only one part per problem. If there are multiple parts turned in for each problem, then it will not be graded and a score of zero will be given for that question.

• Part 3: Critical Thinking and Writing

There are ten critical thinking questions each worth 12 points. Please solve any FIVE questions. It is my hope that these questions will probe your conceptual understanding of the material and see your development in your writing about mathematics.

Additional Instructions

- All work turned in must be NEAT, ORGANIZED, AND STAPLED. Handwriting should be neat and legible.
- Also, it must have a cover page with the following information: the Course Name, Section, Instructor's name (me!), Date Due, name of the assignment and your name all neatly and clearly marked. In addition, write the following statement and sign-your name under it: *"I swear to neither receive nor give any unauthorized assistance on this assignment."*
- It must be written on blank or lined 8.5" by 11" sized paper using only the front side of the page.
- It can be written in pen or pencil. If written in pen, mistakes must be neatly crossed out, or erased with white-out.

- Be sure to label each problem carefully and staple work sequentially.
- Be sure to show all work on every problem.
- The level of "polish" should be very high, just like it would be when turning in an essay in an English class. You should work out solutions on scratch paper before you write your final draft. When applicable, write in complete sentences.
- This assignment is open book and open notes (from seminar, RPs, or previous HHWs), however, you are to work entirely on your own (or you may ask me questions). Consulting other people, textbooks or the internet is strictly forbidden.

DUE THURSDAY, OCTOBER 20, 8:59 PM

You can slide it under my door at any time before the above deadline. Unless for extra-ordinary reasons, no late assignments will be counted/graded.

PART 1 – TRUE or FALSE?

Please spell out in the entire word (TRUE or FALSE) your answer (please don't write T or F only).

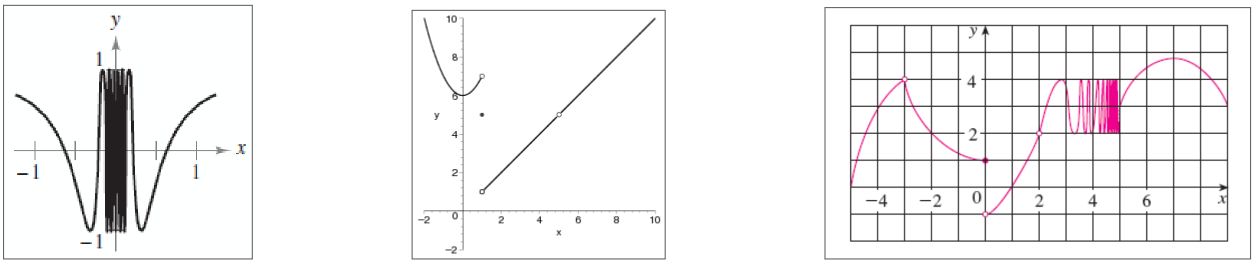

Figure 1: (a) $y = \cos(1/x)$ (b) $y = f(x)$ (c) $y = g(x)$

______ (i) $\lim_{x \to 0} \cos(1/x) =$ DNE because of oscillation (See Figure 1(a))

______ (ii) The function $f(x)$ has a removable discontinuity at $x = 1$ (See Figure 1(b))

______ (iii) $g(2) = 2$ (See Figure 1(c))

______ (iv) $\lim_{x \to 2} g(x) = 2$ (See Figure 1(c))

______ (v) $\lim_{x \to 5^-} g(x) = 3$ (See Figure 1(c))

______ (vi) $\frac{d}{dx}\left[\frac{4x^3 - 12x^2 + 5x}{x}\right] = 8x - 12.$

______ (vii) The function $h(u) = \frac{2u^2}{u^2 + 7}$ is continuous AND differentiable everywhere

______ (viii) $\lim_{t \to 0} \frac{\sin(5t)}{t} = \frac{1}{5}.$

______ (ix) The (instentaneous) velocity function of $s(t) = \frac{\cos(t) - 1}{\cos(t) - 1}$ is defined everywhere

______ (x) $\lim_{x \to 0^-} \left(-\frac{5}{x^7}\right) = +\infty.$

______ (xi) If H is continuous on $[a, b]$ then $\lim_{x \to a^+} H(x) = H(a)$.

______ (xii) Let $a > 1$ and $f(x) = a^x$. Then the equation $f(x) = 0$ has no solutions.

______ (xiii) $\lim_{x \to \pi} \frac{x^2 - \pi^2}{x - \pi} = 2\pi$

______ (xiv) The acceleration function $a(t)$ is the second derivative of the velocity function.

______ (xv) If $0 \le p(x) \le x^3$, then $\lim_{x \to 0} p(x) = 0$.

______ (xvi) The function $y = e^x - |x|$ is continuous AND differentiable everywhere

______ (xvii) $\lim_{y \to -2} \dfrac{2 - 2\cos(y+2)}{y+2} = 2.$

______ (xviii) If $L(x)$ is a linear function, then $L'''(x) = 0$.

______ (xix) A tangent line T at a point P can never cross the graph $y = f(x)$ at a point Q different from P.

______ (xx) One definition of the irrational number "e" is $\lim_{h \to 0} (1+h)^{1/h}$.

PART 2 – A, B, or C?

Problem 1. Limits

(A) [12 points] Evaluate: $\lim_{x \to \pi} \left(\dfrac{e^{x-\pi} - 1}{1 + \ln(x)} - \dfrac{\sin(x)}{\pi x} - \dfrac{2\pi - \sqrt{x^2 + 3\pi^2}}{x^2 - \pi^2} \right) =$

(B) [8 points] Evaluate: $\lim_{x \to 0} \left(\dfrac{\sin(5x)}{x} - \dfrac{\sqrt{9x+64}}{x+1} \right) =$

(C) [4 points] Evaluate: $\lim_{x \to -1} (x^3 + 3x^2 - x - 1) =$

Problem 2. Continuity

(A) [12 points] Is f continuous at $c = e$? Here $f(x) = \begin{cases} x - \cos(\ln(x^{-\pi})), & x < e \\ \frac{1}{3}(7e + 3 - 4e), & x = e \\ \frac{x^2-1}{x-1}, & x > e \end{cases}$

(B) [8 points] For what value of the constant a is the function g continuous on $(-\infty, \infty)$ if $g(x) = \begin{cases} ax^2 - 2x + 3, & x < 1 \\ 6x + 1, & x \geq 1 \end{cases}$

(C) [4 points] Is h continuous at $c = 0$? Here $h(x) = \begin{cases} 4 - 3x^2, & x < 0 \\ 4, & x = 0 \\ \sqrt{16 - x^2}, & x > 0 \end{cases}$

Problem 3. Infinite Limits

(A) [12 points] Evaluate: $\lim_{x \to \pi/2^-} \left(\sqrt{\frac{\tan(x)}{\pi - 4x}} \right)$

(B) [8 points] Evaluate: $\lim_{x \to 2^+} \left(\frac{x+1}{x^2-4} - \ln(x-2) \right)$

(C) [4 points] Evaluate: $\lim_{x \to 0^-} \left(-\frac{5}{x^7} + \frac{x-2}{x^3} \right)$

Problem 4. Squeeze Theorem

(A) [12 points] Show that $\lim_{x \to -\infty} \left(\frac{x^2(\sin(2x) + \cos^2(x))}{(x^2+1)(x-3)} \right) = 0$ using the Squeeze Theorem.

(B) [8 points] Show that $\lim_{x \to +\infty} \left(\frac{\sin(x)}{x} \right) = 0$ using the Squeeze Theorem.

(C) [4 points] Show that $\lim_{x \to 0} \left(x^2 \sin\left(\frac{1}{x} \right) \right) = 0$ using the Squeeze Theorem.

Problem 5. Asymptotes

(A) [12 points] Find all the vertical and horizontal asymptotes of $h(x) = \frac{3x}{\sqrt{x^2-5}}$.

(B) [8 points] Find all the vertical and horizontal asymptotes of $h(x) = \frac{2x^2+4x-6}{x^2+3x-4}$.

(C) [4 points] Find all the vertical and horizontal asymptotes of $h(x) = \frac{x^2+x-6}{x^2+3x-4}$.

Problem 6. Rates of Change

(A) [12 points] If the position function of a moving particle is given by $s(t) = \frac{1}{1+t} + \sqrt{t}$, find the (instantaneous) rate of change of s with respect to time t (also called the velocity function) directly from the definition (i.e. the limit process).

(B) [8 points] If the position function of a moving particle is given by $s(t) = 5\sqrt{t}$, find the (instantaneous) rate of change of s with respect to time t (also called the velocity function) directly from the definition (i.e. the limit process).

(C) [4 points] If the position function of a moving particle is given by $s(t) = 2t^2$, find the (instantaneous) rate of change of s with respect to time t (also called the velocity function) directly from the definition (i.e. the limit process).

Problem 7. Derivatives

(A) [12 points] Evaluate: $\frac{d^2}{dp^2}\left[\frac{1+p}{e^p}\right]$

(B) [8 points] Evaluate: $\frac{d}{du}\left[\frac{\csc(u)}{5u^3}\right]$

(C) [4 points] Evaluate: $\frac{d}{dx}\left[\cos(x)(e^x + 5x^3 - 11x - 3)\right]$

Problem 8. Tangent Lines

(A) [12 points] Find the constants a, b, c so that the parabola $y = ax^2 + bx + c$ passes through the point $(-3, 4)$ and is tangent to the line $y = 2x$ at $(1, 1)$.

(B) [8 points] Find the equation of the line T that is tangent to the graph of $y = g(x) = \frac{-3}{\sec(x) + e^x + 1}$ at the point where $c = 0$.

(C) [4 points] Find the equation of the line T that is tangent to the graph of $y = h(x) = -2e^x(\sin(x) + 3x + 1)$ at the point $P = (0, -2)$.

Problem 9. Applications

(A) [12 points] Let $s(t), v(t), a(t)$ be the position, velocity, and acceleration functions of a moving object. The third and forth derivatives of the position function are also useful for engineering. The *Jerk*, $J(t)$, is the third derivative of position, or the rate of change of the acceleration function. The *Snap*, $S(t)$, is the fourth derivative of position, or the rate of change of the jerk function. In engineering, typically one wants to minimize the jerk function for automobiles or elevators whereas one wants to maximize the jerk function for amusement rides or sports cars. Assume that an elevator's position function is modeled by
$s(t) = \dfrac{t^5}{60} - \dfrac{t^4}{12} + \dfrac{t^3}{3} + \dfrac{t^2}{2} + t$, t minutes after it begins its upward accent.

(i) Find the (instentaneous) velocity, acceleration, jerk, and snap functions for the elevator's position function.

(ii) At what time is the elevator operating at it's smoothest? That is, when is the jerk function minimized?

(iii) What is the snap of the elevator at the time found in part (ii)?

(iv) Explain the significance of your answer in part (iii).

(B) [8 points] The price-demand function for a popular e-book is modeled by
$D(p) = \dfrac{100,000}{p^2 + 10p + 50}$, for $5 \leq p \leq 20$, where $D = D(p)$ is the quantity demanded at the price p dollars.

(i) $(M \to E)$ Explain in words what $\frac{dD}{dp} = D'(p)$ means. [Hint: use the IROC interpretation]

(ii) Compute $\frac{dD}{dp} = D'(p)$.

(iii) Find $D'(5)$, $D'(10)$, and $D'(15)$.

(iv) Interpret the results found in (b). Explain your findings in complete sentences $(M \to E)$.

(C) [4 points] Metalhead Moe fires a baseball upwards off the top of his 6 foot DIY catapult with an initial velocity of 80 feet per second. The distance s (measured in feet) of the baseball from the ground after t seconds is modeled by $s(t) = -16t^2 + v_0 t + s_0$.

(i) What are the equation of motion for the baseball? That is, find $s(t), v(t), a(t)$.

(ii) After how many seconds is the baseball reached it's maximum height? What is the maximum height?

(iii) How long is the baseball in the air?

(iv) What is the impact velocity? That is, the velocity of the baseball when it hits the ground.

Problem 10. Proofs

(A) [12 points] Prove that $\frac{d}{dx}[\cos(x)] = -\sin(x)$ directly the definition of a derivative (ie limit process). [Hint: you'll find the trigonometric formula $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ usefull]

(B) [8 points] Verify that $\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$ by using the derivative rule for sine and the quotient rule.

(C) [4 points] Verify that $\frac{d}{dx}[\cot(x)] = -\csc^2(x)$ by using the derivative rules for sine, cosine, and the quotient rule.

Part 3 - Critical Thinking and Writing

Below are 10 Questions, each worth 12 points. Please select any FIVE questions and present their solutions.

Question 1. About a limit

The statement: "Whether or not $\lim_{x\to a} f(x)$ exists depends on how $f(a)$ is defined" is TRUE:

(a) sometimes, (b) always, (c) never

Justify your answer in complete sentences.

Question 2. Derivatives in the dark

We know $f(1) = 1$ and $f'(1) = 3$. Then $\frac{d}{dx}\left[\frac{f(x)}{x^2}\right]$ when $x = 1$ equals

Question 3. Get Sherlock on the case

You're trying to guess $\lim_{x\to 0^+} f(x)$. You plug-in $x = 0.1, 0.01, 0.001, \ldots$ and get $f(x) = 0$ for each of these values. In fact, you're told that for all $n = 1, 2, 3, \ldots$ we have $f(\frac{1}{10^n}) = 0$. **True or False**: Since the sequence $0.1, 0.01, 0.001, \ldots$ goes to zero, we know that $\lim_{x\to 0} f(x) = 0$. *Explain your reasoning. You can draw a pictures if you can't think of specific examples.*

Question 4. Tangent Lines

The line tangent to the graph of $f(x) = x$ at $P = (0, 0)$

(a) is $y = 0$ (b) is $y = x$ (c) does not exist (d) is not unique. There are infinitely many tangent lines.

Explain your reasoning in complete sentences.

Question 5. Space travel is cool

According to Einstein's (special) theory of relativity, the mass m of a particle depends on it's velocity v according to the formula

$$m = \frac{m_0}{\sqrt{1 - (v/c)^2}},$$

where m_0 is the mass of the object at rest (i.e. when $v = 0$) and c is the speed of light. Find the limit of the mass as the speed v approaches the speed of light c (from the left, i.e. increases to c). Explain the consequences of your discovery for space travelers.

Question 6. A circle can change

If the radius of a circle increases from r_1 to r_2 then the *average* rate of change of the *area* of the circle is

(A) less than $2\pi r_2$

(B) greater than $2\pi r_1$

(C) equal to $2\pi \frac{r_1+r_2}{2}$

(D) all of the above

(E) none of the above

Question 7. Waves & Loops

If $f(x) = \sin(x)$, then

(A) $f(x) = f''''(x)$

(B) $f(x) = -f''(x)$

(C) $f'(x) = \cos(x)$

(D) all of the above.

(E) none of the above.

Question 8. Think in pictures

Consider the function

$$f(x) = \begin{cases} x^2, & x \text{ is rational number} \\ -x^2, & x \text{ is an irrational number} \end{cases}$$

Does $f'(0)$ exist? Explain why or why not.

Question 9. Tangent Lines, Approximate!

If $e^{0.5}$ is approximated by using the tangent line to the graph of $f(x) = e^x$ at $(0, 1)$, and we know $f'(0) = 1$, the approximation is

(A) 0.5

(B) $1 + e^{0.5}$

(C) $1 + 0.5$
(D) all of the above

(E) none of the above

Question 10. Absolutely.

Does the function $f(x) = x|x|$ have a derivative at $x = 1$?