PREREQUISITE/PRE-CALCULUS REVIEW

Introduction

This review sheet is a summary of most of the main topics that you should already be familiar with from your pre-calculus and trigonometry course(s), and which I expect you to know how to use and be able to follow if I use them in passing. If there are topics with which you don't feel as comfortable, suggested homework problems can be found on the web page, and we can also go over them during office hours.

I've tried to include most major topics, but there's a very good chance some things have been overlooked. Feel free to ask me if you don't see a concept listed and are wondering if it will come up in the class.

Coordinate Geometry

What you need to know:

(1) Notation for intervals, and how to plot them on a number line.
(2) Properties/rules of inequalities.
(3) How to plot (x, y) coordinates.
(4) Distance Formula: The distance between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in the xy-plane is

$$|P_1 P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

(5) Equation of a Circle: An equation of the circle with center (h, k) and radius r is

$$(x - h)^2 - (y - k)^2 = r^2$$

[Note, if you take the square root of both sides of this equation, you get the distance formula: the distance between the center (h, k) and a point (x, y) on the circle is r.] In particular, if the center is at the origin $(0, 0)$, this equation becomes

$$x^2 + y^2 = r^2$$

(6) Completing the Square: This comes up when dealing with circles and other conic sections, both in this class and later in Calculus 3. Example 2 on page A17 shows how this can come up in a problem.
(7) Slope of a Line: The slope of a nonvertical line passing through the points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}}$$

(8) The Equation of a Line:
   (a) Point-Slope Form: An equation of the line with slope m that passes through the point (x_1, y_1) is

$$y - y_1 = m(x - x_1)$$

   (b) Slope-Intercept Form: An equation of the line with slope m and y-intercept b is

$$y = mx + b$$

   (you often have to use a known point on the line to solve for b).
   These can be used interchangeably, but the point-slope form tends to arise more naturally in many of the problems we'll be looking at (although we will still solve for y at the end of the problem).
(9) Parallel and Perpendicular Lines:
   (a) Two nonvertical lines are said to be *parallel* if and only if they have the same slope

(b) Two lines with slopes m_1 and m_2 are said to be *perpendicular* if and only if $m_1 m_2 = -1$, i.e., if their slopes are negative reciprocals:

$$m_2 = -\frac{1}{m_1} \quad \text{or, equivalently,} \quad m_1 = -\frac{1}{m_2}$$

(10) How to sketch regions in the plane.

Trigonometry

What you need to know:

(1) Converting from radians to degrees (probably won't use this much):

$$\pi \text{ rad} = 180°, \text{ so } 1 \text{ rad} = \frac{180°}{\pi} \text{ and } 1° = \frac{\pi}{180} \text{ rad}$$

(2) The basic definitions of trigonometric functions in terms of the sides of a right triangle:

$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}} \qquad \cos(\theta) = \frac{\text{adj}}{\text{hyp}} \qquad \tan(\theta) = \frac{\text{opp}}{\text{adj}}$$

$$\csc(\theta) = \frac{\text{hyp}}{\text{opp}} \qquad \sec(\theta) = \frac{\text{hyp}}{\text{adj}} \qquad \cot(\theta) = \frac{\text{adj}}{\text{opp}}$$

(3) The notation for the inverse trigonometric functions (and what these functions are):

$$\arcsin(\theta) \text{ or } \sin^{-1}(\theta), \qquad \arccos(\theta) \text{ or } \cos^{-1}(\theta), \qquad \arctan(\theta) \text{ or } \tan^{-1}(\theta)$$

Remember that this the $^{-1}$ is *inverse* notation, not exponential notation, i.e.,

$$\sin^{-1}(\theta) \neq \frac{1}{\sin(\theta)}, \ \cos^{-1}(\theta) \neq \frac{1}{\cos(\theta)}, \text{ and } \tan^{-1}(\theta) \neq \frac{1}{\tan(\theta)}$$

(4) How to find the values of all trigonometric functions at 'common' angles using either the unit circle (see Figure 1), or by using a $30° - 60° - 90°$ or $45° - 45° - 90°$ triangle (drawn the in the appropriate quadrant).
(5) How to find 'common' values for $\arctan(x)$ (similar to previous item).
(6) The graphs of the four trig functions shown in Figure 2:
(7) The Reciprocal and Quotient Identities:

$$\csc(\theta) = \frac{1}{\sin(\theta)}, \quad \sec(\theta) = \frac{1}{\cos(\theta)}, \quad \cot(\theta) = \frac{1}{\tan(\theta)}, \quad \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}, \quad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

(8) The Pythagorean Identities:

$$1 = \sin^2(\theta) + \cos^2(\theta), \quad \csc^2(\theta) = 1 + \cot^2(\theta), \quad \sec^2(\theta) = \tan^2(\theta) + 1$$

(9) That sine is an odd function and that cosine is an even function, i.e., that

$$\sin(-\theta) = -\sin(\theta) \quad \text{and} \quad \cos(-\theta) = \cos(\theta)$$

(10) The Half-Angle Formulas:

$$\sin(2x) = 2\sin(x)\cos(x)$$
$$\cos(2x) = \cos^2(x) - \sin^2(x)$$
$$= 2\cos^2(x) - 1$$
$$= 1 - 2\sin^2(x)$$

(The last two equalities for $\cos(2x)$ come from using the Pythagorean Identities.)

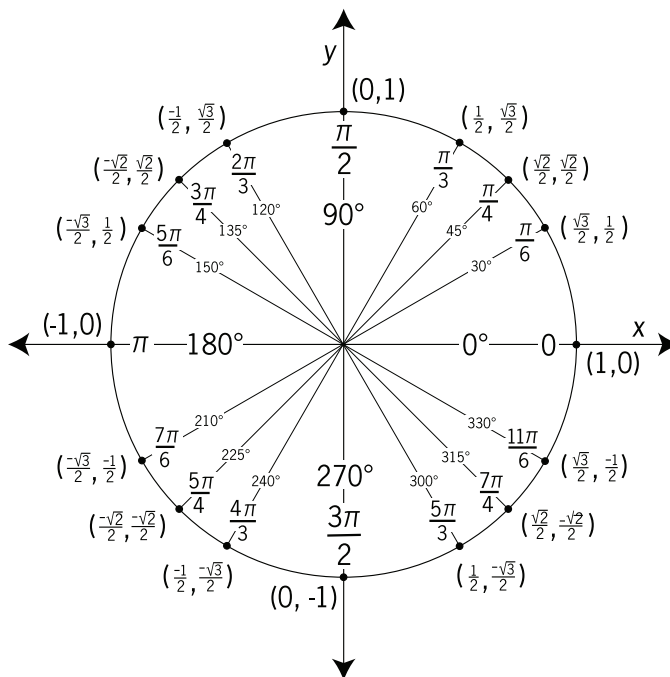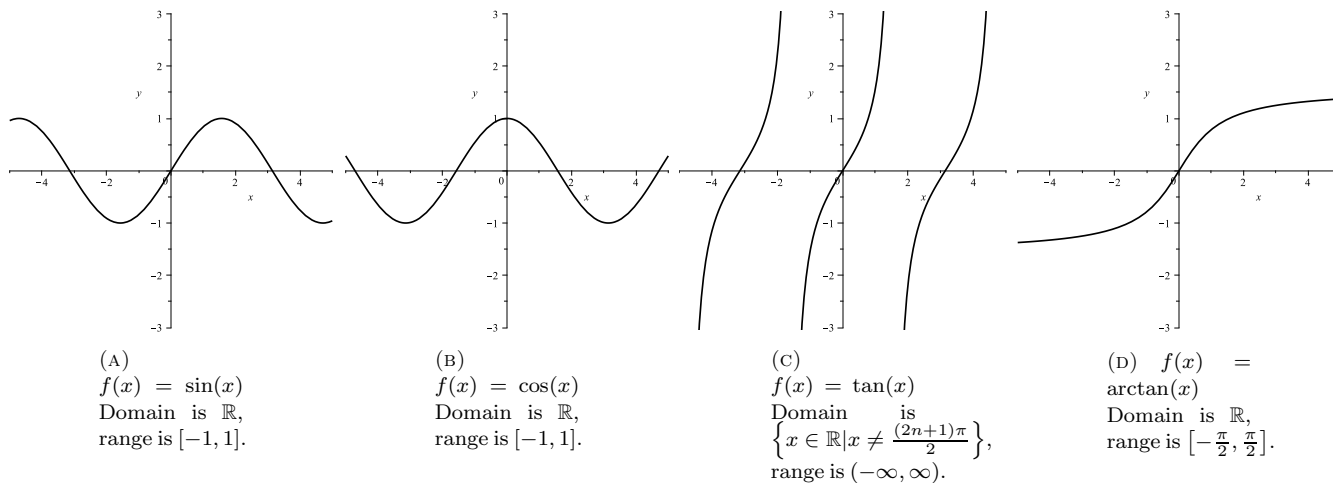FIGURE 1. The Unit Circle (image courtesy FCIT, http://etc.usf.edu/clipart)

(A) $f(x) = \sin(x)$
Domain is $\mathbb{R}$, range is $[-1, 1]$.

(B) $f(x) = \cos(x)$
Domain is $\mathbb{R}$, range is $[-1, 1]$.

(C) $f(x) = \tan(x)$
Domain is $\left\{ x \in \mathbb{R} | x \neq \frac{(2n+1)\pi}{2} \right\}$, range is $(-\infty, \infty)$.

(D) $f(x) = \arctan(x)$
Domain is $\mathbb{R}$, range is $\left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$.

FIGURE 2. Graphs of Trigonometric Functions You Should Know

(11) The Double-Angle Formulas:

$$\cos^2(x) = \frac{1}{2} \left(1 + \cos(2x)\right) = \frac{1 + \cos(2x)}{2}$$
$$\sin^2(x) = \frac{1}{2} \left(1 - \cos(2x)\right) = \frac{1 - \cos(2x)}{2}$$

(12) These facts about sine and cosine:

$$-1 \leqslant \sin(x) \leqslant 1 \quad \text{and} \quad -1 \leqslant \cos(x) \leqslant 1$$
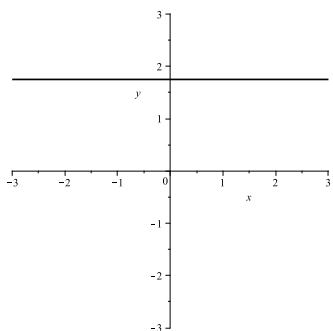
and so

$$|\sin(x)| \leqslant 1 \quad \text{and} \quad |\cos(x)| \leqslant 1$$

We may occasionally use the various addition and subtraction formulas involving trigonometric functions, but we'll reference the book for those as needed.
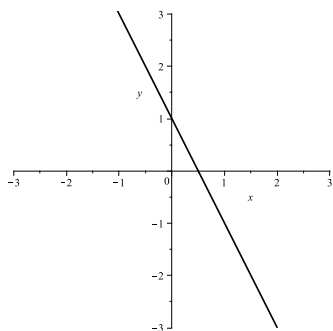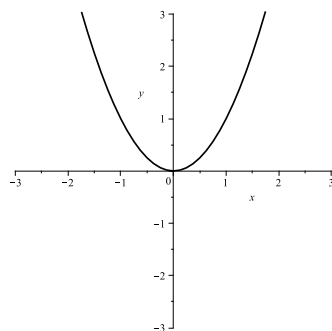
Functions

What you need to know:
(1) The Vertical Line Test (see page 15 in the book).
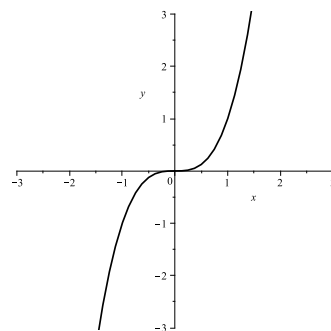(2) The graphs of the functions shown in Figure 3:

(A) Constant, $f(x) = a$ ($a = 1.75$ shown). Domain is $\mathbb{R}$, range is b.
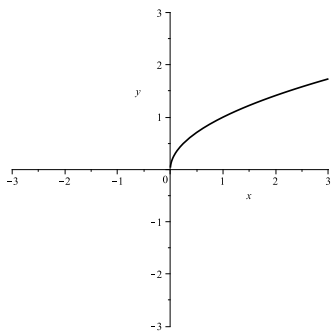
(B) Linear, $f(x) = ax + b$ ($y = -2x + 1$ shown). Domain is $\mathbb{R}$, range is $\mathbb{R}$.
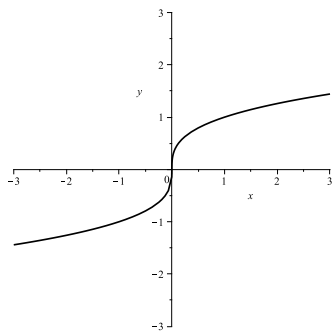
(C) Square, $f(x) = x^2$ Domain is $\mathbb{R}$, range is $[0, \infty)$.

(D) Cubic, $f(x) = x^3$ Domain is $\mathbb{R}$, range is $\mathbb{R}$.

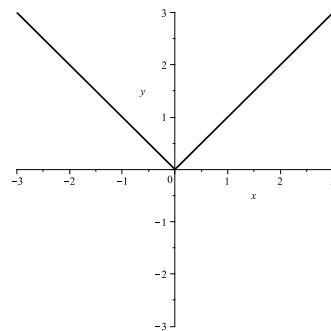(E) Square-root, $f(x) = \sqrt{x}$ Domain is $[0, \infty)$, range is $[0, \infty)$.

(F) Cube-root, $f(x) = \sqrt[3]{x}$ Domain is $\mathbb{R}$, range is $\mathbb{R}$.
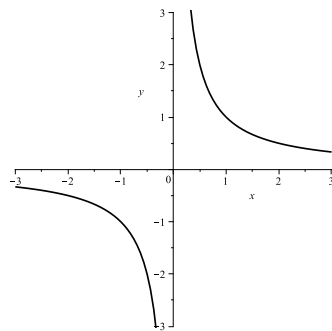
(G) Reciprocal, $f(x) = \frac{1}{x}$ Domain is $(-\infty, 0) \cup (0, \infty)$, range is $(-\infty, 0) \cup (0, \infty)$.

(H) Absolute value, $f(x) = |x|$ Domain is $\mathbb{R}$, range is $[0, \infty)$.

FIGURE 3. Graphs of Basic Functions You Should Know

(3) What a Difference Quotient of a function $f(x)$ is (and how to compute/simplify them):
$$\frac{f(x+h) - f(x)}{h} \quad \text{or} \quad \frac{f(x) - f(a)}{x - a}$$

(4) How to plot piecewise defined functions.

(5) What it means for a function to be even ($f(-x) = f(x)$) or odd ($f(-x) = -f(x)$) - these will be used occasionally.

(6) What it means for a function to be increasing or decreasing (this will come up a lot): Given a function $f(x)$ and some interval I,
- $f(x)$ is said to be *increasing* on I if $f(x_1) < f(x_2)$ for all $x_1 < x_2 \in I$.
- $f(x)$ is said to be *decreasing* on I if $f(x_1) > f(x_2)$ for all $x_1 < x_2 \in I$.

(7) Basics of polynomials:
- What a root or zero is,
- the Quadratic Equation (for finding the roots/zeros of $y = ax^2 + bx + c$):
$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
- what we mean by degree,
- how the degree and leading coefficient effect end behavior.

(8) How to plot quadratics (such as those in Figure 4) by factoring or completing the square.
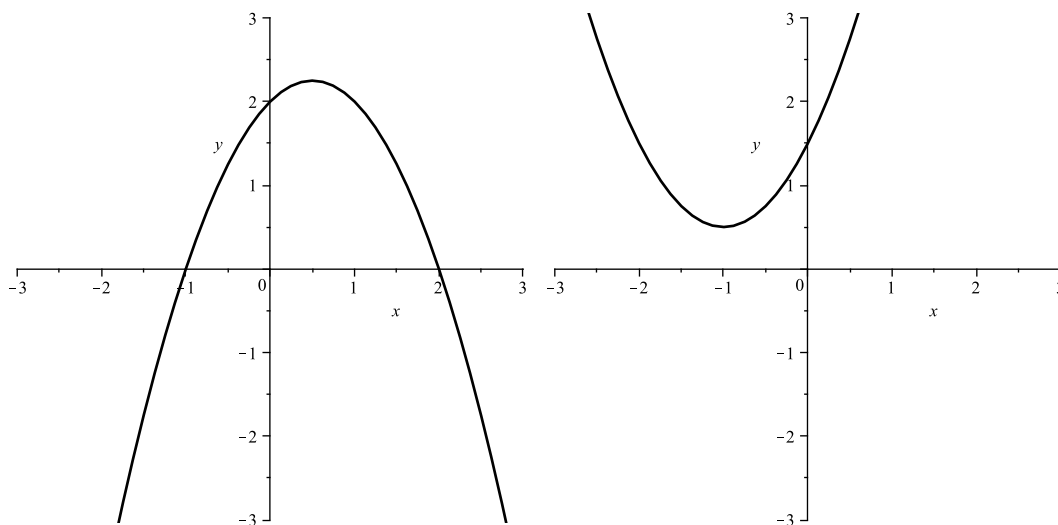
(A) Factoring: $y = -x^2 + x + 2 \to y = -(x^2 - x - 2) \to y = -(x + 1)(x - 2)$, so we have a parabola opening downward with x-intercepts at -1 and 2.

(B) Completing the Square: $y = x^2 + 2x + \frac{3}{2} \to y = (x^2 + 2x + 1) - 1 + \frac{3}{2} \to y = (x+1)^2 + \frac{1}{2}$, so we have the graph of $y = x^2$ shifted left one unit and up $\frac{1}{2}$-unit.

FIGURE 4. Two Examples of Graphing Quadratics

(9) What a rational function is, and how to find its domain.

(10) Given two functions $f(x)$ and $g(x)$, know how to find the composition function $(f \circ g)(x) = f(g(x))$.

Conic Sections

What you need to know:

Equation of a Circle. The equation of a circle with radius r centered at the origin $(0,0)$ is

$$x^2 + y^2 = r^2$$

The equation of a circle with radius r centered at the point (h,k) is

$$(x-h)^2 + (y-k)^2 = r^2$$

Equation of an Ellipse. The equation of an ellipse centered at the origin $(0,0)$ is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The equation of a centered at the point (h,k) is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Equation of a Hyperbola. The equation of a hyperbola "centered" at the origin $(0,0)$ with foci on the x-axis is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The asymptotes are $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$.
    The equation of a hyperbola "centered" at the origin $(0,0)$ with foci on the y-axis is

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

The asymptotes are $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$.
    The equation of a hyperbola "centered" at the point (h,k) is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

or

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

Creating New Functions from Known Functions

You need to know about the various shifts and transformations (shrinking and stretching won't come up very much in this class, so focus on the others):

(1) **Shifting:** Given a function $y = f(x)$ and a constant $c > 0$, we have the following:
   - $y = f(x) + c$ shifts the graph **up** c units (add c to y-values).
   - $y = f(x) - c$ shifts the graph **down** c units (subtract c from y-values).
   - $y = f(x + c)$ shifts the graph **left** c units (subtract c from x-values, i.e., replace x with $x + c$).
   - $y = f(x - c)$ shifts the graph **right** c units (add c to x-values, i.e., replace x with $x - c$).

(2) **Reflecting:**
   - $y = -f(x)$ reflects graph about the x-**axis** (multiply all y values by -1, i.e., multiply function by -1).
   - $y = f(-x)$ reflects graph about the y-**axis** (multiply all x values by -1, i.e., replace x with $-x$).

(3) **Stretching & Shrinking:** Given a function $y = f(x)$ and a constant $c > 1$:
   - $y = c \cdot f(x)$ gives a **vertical stretch** by a factor of c (multiply y-values by c, i.e., multiply function by c).
   - $y = \frac{1}{c} \cdot f(x)$ gives a **vertical shrink** by a factor of $\frac{1}{c}$ (divide y-values by c, i.e., divide function by c).
   - $y = f(c \cdot x)$ gives a **horizontal shrink** by a factor of $\frac{1}{c}$ (divide x-values by c, i.e., replace x with $c \cdot x$ in the function).
   - $y = f(\frac{1}{c} \cdot x)$ gives a **horizontal stretch** by a factor of c (multiply x-values by c, i.e., replace x with $\frac{1}{c} \cdot x$ in the function).

Exponential Functions and Logarithms

Exponential Functions and Laws of Exponents. An exponential function with base a is of the form

$$f(x) = a^x, \quad a > 0 \text{ (and typically } a \neq 1)$$

If s, t, a, b are real numbers with $a > 0$ and $b > 0$, then

$$1^s = 1$$
$$a^0 = 1$$
$$a^s a^t = a^{s+t}$$
$$\left(a^s\right)^t = a^{st} = \left(a^t\right)^s$$
$$(ab)^s = a^s b^s$$
$$\left(\frac{a}{b}\right)^s = \frac{a^s}{b^s}$$
$$a^{-s} = \frac{1}{a^s} = \left(\frac{1}{a}\right)^s$$

The following property is one of the ways we can solve exponential equations or inequalities (you are expected to know how to do this), either by getting a single exponential on each side with the same base and then setting exponents equal, or by taking a number raised to both sides:

$$a^u = a^v \Leftrightarrow u = v$$

Graphs of Exponential Functions. We can take $f(x) = a^x$ and reflect it about the x- or y-axis. For $a > 1$, we get the graphs in Figure 5 (graphs for $0 < a < 1$ can be obtained by taking the reciprocal of the base and multiplying the exponent by -1, e.g., $y = \left(\frac{1}{2}\right)^x$ is the same as $y = 2^{-x}$).

(A) $f(x) = a^x$. Domain is $\mathbb{R}$, range is $(0, \infty)$, points marked are $\left(-1, \frac{1}{a}\right)$, $(0, 1)$, $(1, a)$

(B) $f(x) = -a^x$. Domain is $\mathbb{R}$, range is $(0, -\infty)$, points marked are $\left(-1, -\frac{1}{a}\right)$, $(0, -1)$, $(1, -a)$

(C) $f(x) = a^{-x}$. Domain is $\mathbb{R}$, range is $(0, \infty)$, points marked are $(-1, a)$, $(0, 1)$, $\left(1, \frac{1}{a}\right)$

(D) $f(x) = -a^{-x}$. Domain is $\mathbb{R}$, range is $(0, -\infty)$, points marked are $(-1, -a)$, $(0, -1)$, $\left(1, -\frac{1}{a}\right)$
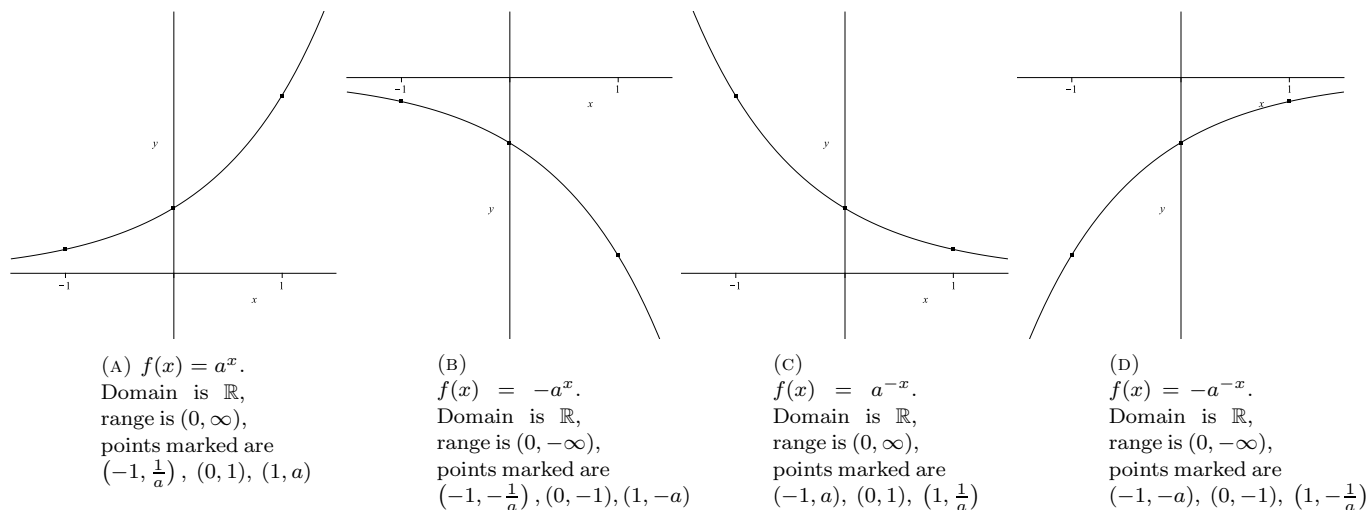
FIGURE 5. Graphs of Exponential Functions (Other shifts/transformations may also be applied.)

Logarithmic Functions and Laws of Logarithms. A logarithmic function looks like:

$$f(x) = \log_a(x), \text{ where } a > 0, a \neq 1$$

(Note: $y = \log_a(x) \Leftrightarrow x = a^y$.)

When the base of a log function is e, we write $\ln(x)$ instead of $\log_e(x)$ $(y = \ln(x) \Leftrightarrow x = e^y)$.

When the base of a log function is 10, we write $\log(x)$ instead of $\log_{10}(x)$ $(y = \log(x) \Leftrightarrow x = 10^y)$.

The following are properties of logarithmic functions (these hold for any base, including e):

$$\log_a(1) = 0$$
$$\log_a(a) = 1$$
$$a^{\log_a(M)} = M$$
$$\log_a(a^r) = r$$
$$\log_a(MN) = \log_a(M) + \log_a(N)$$
$$\log_a\left(\frac{M}{N}\right) = \log_a(M) - \log_a(N)$$
$$\log_a(M^r) = r\log_a(M)$$
$$\log_a(x) = \frac{\ln(x)}{\ln(a)} = \frac{\log(x)}{\log(a)} \text{ (this is known as the Change of Base Formula)}$$

The following property give us a couple of the ways we can solve logarithmic equations or inequalities (you are expected to know how to do this), either by getting a single log with the same base on each side and then setting the arguments equal, or by taking the log of both sides (using an appropriate base):

$$\log_a(M) = \log_a(N) \Leftrightarrow M = N$$

Graphs of Logarithmic Functions. We can take $f(x) = \log_a(x)$ and reflect it about the x- or y-axis. For $a > 1$, we get the graphs in Figure 6 (graphs for $0 < a < 1$ can be obtained by taking the reciprocal of the base and then multiplying by -1, e.g., $y = \log_{\frac{1}{2}}(x)$ is the same as $y = -\log_2(x)$).

Inverse Functions

What you need to know:
1. What it means for a function to be 1-1 (read "one-to-one").
2. The Horizontal Line Test.
3. The notation for compositions of functions $[(f \circ g)(x) = f(g(x))]$.
4. How to find the inverse of a given function (here are the steps):
   (a) Check to that the function is $1 - 1$ (at least on the domain given). If it's not, then either the domain has to be restricted (usually told in the problem that this is the case) or you can't find the inverse.
   (b) Interchange the xs and ys, i.e., instead of $y = f(x)$ now write $x = f(y)$.
   (c) Solve for y. The result will be $f^{-1}(x)$.
5. How the domain and range of f and f^{-1} relate to each other.
6. What happens when you compose a function with its inverse.
7. That the graphs of f and f^{-1} are symmetric about the line $y = x$.

(A) $f(x) = \log_a(x)$. Domain is $(0, \infty)$, range is $\mathbb{R}$, points marked are $\left(\frac{1}{a}, -1\right)$, $(1, 0)$, $(a, 1)$

(B) $f(x) = -\log_a(x)$. Domain is $(0, \infty)$, range is $\mathbb{R}$, points marked are $\left(\frac{1}{a}, 1\right)$, $(1, 0)$, $(a, -1)$

(C) $f(x) = \log_a(-x)$. Domain is $(-\infty, 0)$, range is $\mathbb{R}$, points marked are $(-a, 1)$, $(-1, 0)$, $\left(-\frac{1}{a}, -1\right)$

(D) $f(x) = -\log_a(-x)$. Domain is $(-\infty, 0)$, range is $\mathbb{R}$, points marked are $(-a, -1)$, $(-1, 0)$, $\left(-\frac{1}{a}, 1\right)$
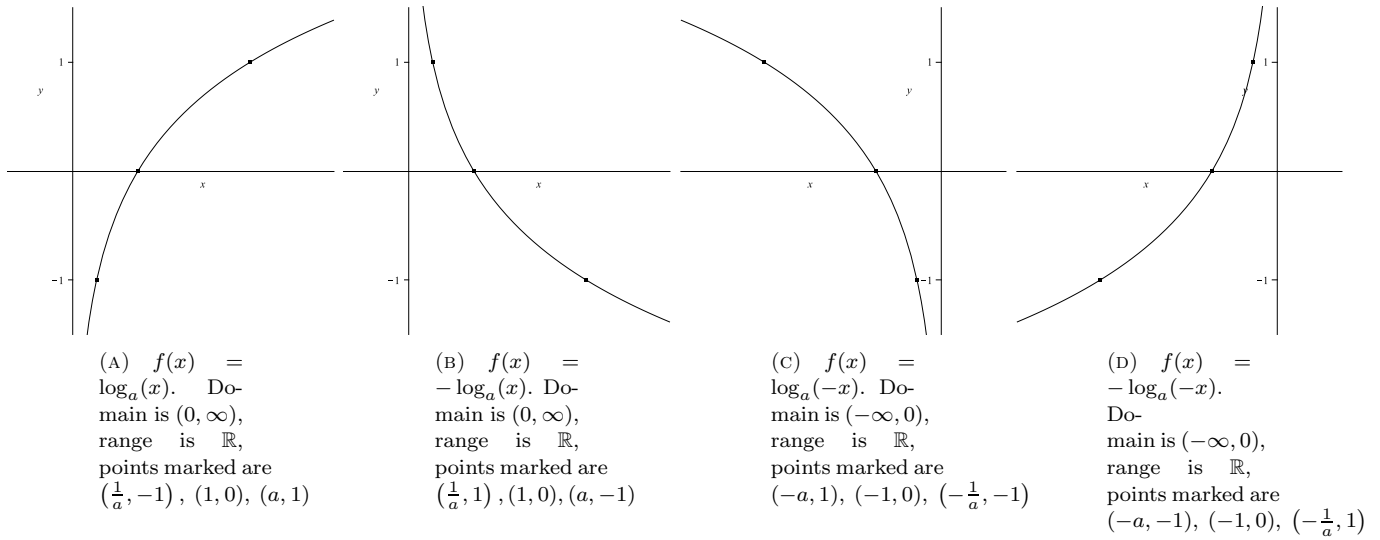
FIGURE 6. Graphs of Logarithmic Functions (Other shifts/transformations may also be applied.)