PITZER COLLEGE

MATH 32: Calculus III — SPRING 2018

Notes — Dr. Basilio

Updated: 5.1.2018

This document contains:

- Exam day info
- Final Exam info
- Exam 1, 2 and 3 topics covered list
- Notes on Chapters: 12, 13, 14, 15, and 16.1-16.4 and bonus material on 16.5-16.9
- Exam 1, 2, 3, and Final Exam Practice Problems
- Exam 1, 2, 3, and Final Exam Practice Problems Answers to even problems

What to expect on Exam day (for Exams 1 and 2)

- I'll arrive to our classroom before 12:50 pm and we'll have a Q&A where I'll answer any questions you have until 1:25 pm. Then you'll bring all of your belongings to the front of the classroom and take Exam 1 from 1:30-2:30pm.
- So, the length of time is 60 minutes. Though I'll usually allow an extra 5-10 minutes if you want/need the time.
- You cannot use any calculator or electronic devise during the exam.
- Once the exam starts you may not use the restroom. So please use the restroom before the exam starts or during the first 30 minutes.
- Expect a mix of True/False, Multiple Choice, and Free Response questions. See the Practice Problems below.

What to expect for Exam 3

- This will be a take-home exam. Details and Exam information will be posted on Sakai.

What to expect for Final Exam

- Monday, May 7 from 2-5 pm in our usual classroom.
- I'll arrive to our classroom before 1:30 pm and we'll have a Q&A where I'll answer any questions you have until 1:55 pm. Then you'll bring all of your belongings to the front of the classroom and take the Final Exam from 2-5pm.

- You cannot use any calculator or electronic devise during the exam.
- Students will get one bathroom break, must turn in their exam while they leave the room, and only one student at a time.
- Expect a mix of True/False, Multiple Choice, and Free Response questions. See the Practice Problems below.
- It is cumulative exam, so everything we covered is fair game. We'll have a slight focus on Chapter 16 material, especially from 16.3 and 16.4, accounting for roughly 40% of the exam. The remaining 60% will be material from Chapter 12, 13, 14, and 15 with a roughly 50-50 split between differentiation and integration.

Material Covered

EXAM 1: Monday, February 12

Chapter 12: Vectors and the Geometry of Space

12.1 - Three-Dimensional Coordinate Systems
12.2 - Vectors
12.3 - The Dot Products
12.4 - The Cross Products
12.5 - Equations of Lines and Planes
12.6 - Cylinders and Quadric Surfaces

Chapter 13: Vectors Functions

13.1 - Vector Functions and Space Curves
13.2 - Derivatives and Integrals of Vector Functions

EXAM 2: March 19

Exam 1 Material

Chapter 13: Vectors Functions

13.3 - Arc Length and Curvature
13.4 - Motion in Space: Velocity and Acceleration

Chapter 14: Partial Derivatives

14.1 - Functions of Several Variables
14.2 - Limits and Continuity
14.3 - Partial Derivatives
14.4 - Tangent Planes and Linear Approximations
14.5 - The Chain Rule
14.6 - Directional Derivatives and the Gradient Vector
14.7 - Maximum and Minimum Values

EXAM 3: due April 30 by 5 pm (Take-home)

Exam 1 & 2 Material

Chapter 14: Partial Derivatives

14.8 - Lagrange Multipliers

Chapter 15: Multiple Integrals

15.1 - Double Integrals over Rectangles
15.2 - Double Integrals over General Regions
15.3 - Double Integrals in Polar Coordinates
15.4 - Applications of Double Integrals
15.6 - Triple Integrals
15.7 - Triple Integrals in Cylindrical Coordinates
15.8 - Triple Integrals in Spherical Coordinates

Chapter 18: Vector Calculus

16.1 - Vector Fields
16.2 - Line Integrals

FINAL EXAM on Monday, May 7 from 2-5 pm

Everything we covered from Chapters 12, 13, 14, 15, and 16 up to 16.4.

Notes

Chapter 12: Vectors and the Geometry of Space

- The length of a vector and the relationship to distances between points
- Addition, subtraction, and scalar multiplication of vectors, together with the geometric interpretations of these operations
- Basic properties of vector operations
- The dot product: $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$
- Basic algebraic properties
- The geometric meaning of the dot product in terms of lengths and angles: in particular the formula $\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos(\theta)$
- Angle formula: $\theta = \cos^{-1}\left(\dfrac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|}\right)$
- $\boxed{\|\vec{a}\|^2 = \vec{a} \cdot \vec{a}}$
- Vector projections: geometric meaning and formulas.
  Projection of $\vec{b}$ onto $\vec{a}$: $comp_{\vec{a}}(\vec{b}) = \dfrac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|}$ this is just a length.
  There is also the vector version that points along the direction of $\vec{a}$:
  $proj_{\vec{a}}(\vec{b}) = \dfrac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|^2}\,\vec{a}$ or $proj_{\vec{a}}(\vec{b}) = \dfrac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}}\,\vec{a}$.
- The cross product: definition and basic properties
- The geometric meaning of the cross product: in particular $\vec{v} \times \vec{w}$ is orthogonal to $\vec{v}$ and $\vec{w}$, with magnitude $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \, \|\vec{w}\| \sin(\theta)$, and direction given by the right-hand rule
- $\|\vec{v} \times \vec{w}\|$ is the area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$.
- $\vec{u} \cdot (\vec{v} \times \vec{w})$ is the volume of the parallelopiped spanned by $\vec{u}$, $\vec{v}$ and $\vec{w}$.
- Tests for Orthogonality:
  - $\vec{v}$ and $\vec{w}$ are orthogonal $\iff$ $\vec{v} \cdot \vec{w} = 0$
  - $\vec{v}$ and $\vec{w}$ are parallel $\iff$ $\vec{v} \times \vec{w} = 0$
  - $\vec{u}$, $\vec{v}$ and $\vec{w}$ are coplanar $\iff$ $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$
- LINES AND PLANES WITH VECTORS
- Intrinsic description (vectors) vs. Extrinsic description (scalar equations)
- Lines: passage between a vector equation, parametric equations, and symmetric equations
- Vector Eq of a line: $\boxed{\vec{r} = \vec{P} + t\vec{v}}$ (in book $\vec{r}_0 = \vec{P}$)
- line segment between two points

- Planes: passage between a vector description (a point together with two direction vectors) and a scalar equation
- Vector Eq of a plane: $\boxed{\vec{n} \cdot \vec{v} = 0}$ (in book $\vec{r} - \vec{r}_0 = \vec{v} = \langle x - x_0, y - y_0, z - z_0 \rangle$)
- Distance from point P and a plane $\mathcal{P} : ax + by + cz + d = 0$: $\boxed{D = comp_{\vec{n}}(\vec{PQ})}$, where Q is any point on $\mathcal{P}$, or $D = \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}$
- Using vector algebra to solve geometric problems about lines and planes–it is essential that you think geometrically and try to save the number crunching in components for the last moment.
- GEOMETRY OF SURFACES
- Cylinders: know how to spot a "free (missing) variable" to help sketch
- QUADRIC SURFACES: Spheres, Cones, Ellipsoids, Elliptic Paraboloid, Hyperboloid of 1-sheet, Hyperboloid of 2-sheets, Hyperbolic Paraboloid
- Be able to recognize the above either by memorizing their equations or by using intersection with planes as done in class

Chapter 13: Vectors Functions

- $\boxed{\text{Functions } f : X \to Y}$
  where set X is domain (=set of inputs), Y is the range (=set of outputs)
- We'll only worry about: $\boxed{f : \mathbb{R}^n \to \mathbb{R}^m}$ with $n, m \geq 1$
- $n = m = 1$: real-valued function of a real variable $f : \mathbb{R} \to \mathbb{R}$
  $x \in \mathbb{R}, y \in \mathbb{R}$, usually written $y = f(x)$
  Graph is a curve in the plane
- When $Y = \mathbb{R}$: scalar-valued functions
- When $X = \mathbb{R}$ and $Y = \mathbb{R}^2$: plane curves or vector-valued functions
  $t \in \mathbb{R}$, $f(t) \in \mathbb{R}^2$ usually written $f(t) = \vec{r}(t) = \langle f(t), g(t) \rangle = f(t)\hat{\imath} + g(t)\hat{\jmath}$
  Graph is a plane curve moving throughout 2D plane
- When $X = \mathbb{R}$ and $Y = \mathbb{R}^3$: space curves or vector-valued functions
  $t \in \mathbb{R}$, $f(t) \in \mathbb{R}^3$ usually written $f(t) = \vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$
  Graph is a space curve moving throughout 3D plane
- Line segment from a point P to Q: $\vec{\sigma}(t) = (1 - t)P + tQ$, $t \in [0, 1]$
- Sketching space curves, vector-valued functions
- Space Curves/VVFs: limits, continuity, differentiation rules (Theorem 3, p. 858), definite integral
- Example 4 on p. 858, know this proof
- Arclength = length of a curve; $\boxed{L = \int_a^b |\vec{r}\,'(t)| dt}$
  Alternatively, you can use: $L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} dt$
- unit tangent vector: $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

- Curvature = bending from flat; $\kappa(t) = \dfrac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \dfrac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'(t)|^3}$

- TNB Frame: $\vec{T}$, $\vec{N}$, $\vec{B}$ all unit length and mutually orthogonal to each other. Hence, making a little "frame":
  $\vec{N}(t) = \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|}$ and $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

- Given a space curve $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, we call $\vec{r}(t)$ the position vector-valued function. The velocity vector-valued function is the derivative of the position function:$\vec{v}(t) = \vec{r}'(t)$ and it's speed is the length of the velocity vector: $|\vec{v}(t)|$. It's acceleration VVF is the derivative of the velocity: $\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$.

- Newton's Second Law: $\vec{F} = m\vec{a}$.

- Vector Differential Equations; initial conditions

Chapter 14: Partial Derivatives

- Functions: $\boxed{f : \mathbb{R}^n \to \mathbb{R}^m}$ with $n, m \geq 1$
  Now, we will have $n > 1$: functions of several variables!

- $n = 2, m = 1$: Scalar-Valued function of TWO variables
  $(x, y) \in \mathbb{R}^2$, $f(x, y) \in \mathbb{R}$
  Graph is $\boxed{z = f(x, y)}$
  Graph is a surface in space
  Domain D is a subset of the plane $\mathbb{R}^2$
  Level Curves: $f(x, y) = k$ for k fixed are curves in plane with height fixed–"isotherms"

- $n > 3, m = 1$: SVFs of three or more variables
  $(x_1, x_2, x_3, \ldots, x_n) \in \mathbb{R}^n$, $f(x_1, x_2, x_3, \ldots, x_n) \in \mathbb{R}$
  Graph: none! Instead need to use other techniques
  Level Surfaces: $f(x_1, x_2, x_3, \ldots, x_n) = k$ for k fixed

- Limits: $\boxed{\lim_{(x,y)\to(a,b)} f(x, y) = L}$ means: "as (x, y) approaches (a, b) along any possible path, the values $f(x, y)$ approach the unique value L."

- Know how to compute limits and to show when limits DNE by using different paths

- Continuity: $\boxed{\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)}$

- Partial Derivatives: Given $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y)$
  $\boxed{\dfrac{\partial f}{\partial x}(a, b) = \lim_{hto0} \dfrac{f(a + h, b) - f(a, b)}{h}}$ the partial derivative of f with respect to x at the point (a, b)
  $\boxed{\dfrac{\partial f}{\partial y}(a, b) = \lim_{hto0} \dfrac{f(a, b + h) - f(a, b)}{h}}$ the partial derivative of f with respect to y at the point (a, b)
  BUT: computing them is easy! Just: "pretend the other variable is constant"

- Know the geometry of the partial derivatives as slopes of the appropriate tangent lines

- Implicit Diff with partial derivatives

- Higher partial derivatives: $f_{xx} = \frac{\partial^2 f}{\partial x^2}$, etc

- Clairaut's Theorem: equality of mixed partials is when the second-order partial derivatives are continuous functions

- Tangent Planes: Given $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y)$
  The tangent plane of f at $P = (a, b, f(a, b))$ is $\boxed{z = f(a, b) + f_x(a, b) \cdot (x - a) + f_y(a, b) \cdot (y - b)}$
  Know how this formula was derived in class with $\vec{n} = \langle -f_x, -f_y, 1 \rangle$

- Linearization: $\boxed{L(x, y) = f(a, b) + f_x(a, b) \cdot (x - a) + f_y(a, b) \cdot (y - b)}$
  When (x, y) is close to (a, b), then $f(x, y) \approx L(x, y)$–that is the linearization is a good approximation of f near P

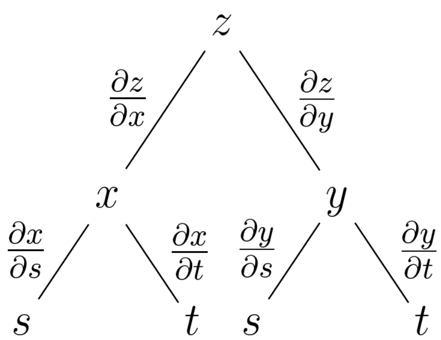
- f is differentiable at $P = (a, b, f(a, b))$ if the tangent plane exists at P.
  Notice: this is stronger than simply requiring that the partial derivatives f_x and f_y exist at P.
  Theorem: if f_x and f_y are continuous, then f is differentiable

- Differentials:
  dx and dy can be any real numbers (usually, $dx = \Delta x = x_2 - x_1$, $dy = \Delta y = y_2 - y_1$)
  Actual change in $z = f(x, y)$ from $P = (x_1, y_1)$ to $Q = (x_2, y_2)$ is: $\boxed{\Delta z = z_2 - z_1 = f(Q) - f(P)}$
  Approximate change is given by the differential dz: $\boxed{dz = f_x(a, b) \cdot dx + f_y(a, b) \cdot dy}$
  dz sometimes called the total differential
  Works for higher-dimensions too: $dz = f_{x_1} \cdot dx_1 + f_{x_2} \cdot dx_2 + \cdots + f_{x_n} \cdot dx_n$

- Chain Rule:
  Basic chain rule: $f : \mathbb{R}^3 \to \mathbb{R}$ with $f(x, yz)$, $g(t) : \mathbb{R} \to \mathbb{R}^3$ with $g(t) = \langle x(t), y(t), z(t) \rangle$, then the derivative of $(f \circ g)(t) : \mathbb{R} \to \mathbb{R}$ is

$$\frac{d}{dt} f(x(t), y(t), z(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

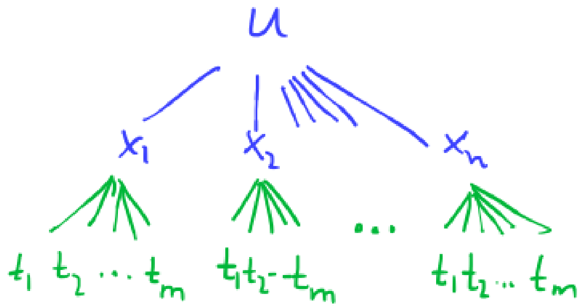
  Tree diagrams are helpful for book-keeping:



- General Chain Rule:
  Assume $u : \mathbb{R}^n \to \mathbb{R}$ is a SVF of n variables written $u(x_1, x_2, \ldots, x_n)$ and each $x_i : \mathbb{R}^m \to \mathbb{R}$ is a SVF of m variables written $x_i(t_1, t_2, \ldots, t_m)$ for each $i = 1, 2, \ldots n$. Then

$$\frac{\partial u}{\partial t_j} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_j}$$

  Notice: in the above formula the t_j is the same, but we take all possible partial derivatives of u with respect to the x_i's as i ranges from 1 to n. The tree diagram is helpful:



- Gradient Vector: Given $f(x,y)$ or $f(x,y,z)$ the gradient collects all the partial derivatives into a vector:
  $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle$ or $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle$
  Common notations: $\nabla f = \text{grad}(f) = \text{del}(f) = \partial(f)$
  This generalizes easily to higher dimensions

- Directional Derivative:
  The directional derivative of f in the direction of the unit vector $\vec{u} = \langle u_1, u_2\rangle$ (or $\vec{u} = \langle u_1, u_2, u_3\rangle$):
  $\boxed{D_{\vec{u}}(f) = f_x(a,b)\cdot u_1 + f_y(a,b)\cdot u_2}$ or $\boxed{D_{\vec{u}}(f) = f_x(a,b,c)\cdot u_1 + f_y(a,b,c)\cdot u_2 + f_z(a,b,c)\cdot u_3}$
  This generalizes easily to higher dimensions. We can write it compactly for all dimensions as: $\boxed{D_{\vec{u}}(f) = \nabla(f)\cdot\vec{u}}$

- Maximizing the Directional derivative:
  the maximum of $D_{\vec{u}}(f)$ at a point $P = (a,b)$ is given by $|\nabla f(a,b)|$ and occurs when $\vec{u}$ is in the same direction as $\nabla f(a,b)$.
  the minimum of $D_{\vec{u}}(f)$ at a point $P = (a,b)$ is given by $-|\nabla f(a,b)|$ and occurs when $\vec{u}$ is in the opposite direction as $\nabla f(a,b)$.

- Level Surfaces, Tangent Planes, and Gradients
  Given a function $F : \mathbb{R}^3 \to \mathbb{R}$. Consider it's level surface $S : F(x,y,z) = k$. Then the gradient of F is normal to the tangent plane at a point $P = (a,b,c)$ on the surface S (as long as it's not the zero vector), that is

  $$(\nabla F)(a,b,c,)\cdot\vec{r}\,'(t_0) = 0$$

  for any space curve $\vec{r}(t)$ that travels inside the surface S and passes through P at t_0.
  We can use this to find the equation of the tangent plane: $(\nabla F)(a,b,c)\cdot\langle x-a, y-b, z-c\rangle = 0$.

  BONUS MATERIAL
  How is this related to the derivation of the tangent plane we learned earlier?
  Previously we started with $z = f(x,y)$ a function of two variables and its graph was a surface S.
  We can view it as a function of three variables $F(x,y,z) = z - f(x,y)$ and the surface S is the level surface of F with $k = 0$.
  From the gradient equation for $F(x,y,z) = z - f(x,y)$:

  $$\begin{aligned}\nabla F(x,y,z) &= \langle\frac{\partial}{\partial x}(z - f(x,y)), \frac{\partial}{\partial y}(z - f(x,y)), \frac{\partial}{\partial z}(z - f(x,y))\rangle \\ &= \langle -f_x(x,y), -f_y(x,y), 1\rangle\end{aligned}$$

  This was exactly what we got in section 14.4 where we used $\vec{n} = \vec{f_x}\times\vec{f_y} = \langle 1,0,f_x\rangle\times\langle 0,1,f_y\rangle$.

- MAX & MIN VALUES: know the definitions of a local min/local max and global min/global max VALUES of a function f.
  Know the distinction between the min/max value of f and the point where it occurs.

- Critical Points: $P = (a, b)$ is a critical point of f if $\nabla f(a, b) = 0$ or DNE. That is, if $f_x(a, b) = 0$ and $f_y(a, b) = 0$; or if one of f_x or f_x DNE.

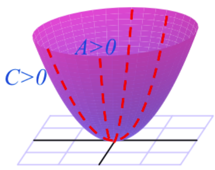
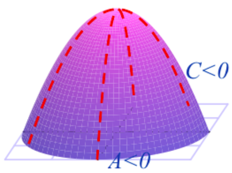
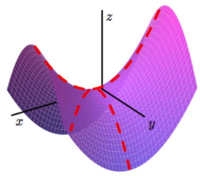
- "Fermat's Theoem:" If f has a local min/max at P and f is differentiable at P, then P is a critical point of f

- C^2 functions = second-order partial derivatives exist and are continuous

- Know: Let $A = f_{xx}(a, b)$, $C = f_{yy}(a, b)$, $B = f_{xy}(a, b)$.
  Let $\boxed{D = AC - B^2}$ called the discriminant.

- SDT: Second Derivative Test:
  Assume: f is C^2 and $P = (a, b)$ is a critical point of f.

Second Derivative Test

• if $D > 0$ and $A > 0$ then $f(a, b)$ is a local MIN value	if $D > 0$ and $A < 0$ then $f(a, b)$ is a local MAX value	if $D < 0$ then $f(a, b)$ is NOT an extremum (saddle point)	if $D = 0$ then test fails (anything can happen)
			

Note: when $D > 0$, then $AC - B^2 > 0$ so $AC > B^2 > 0$. This implies that both A and C have the same sign. So either both $A > 0$ and $C > 0$ or both $A < 0$ and $C < 0$. This is why the bending in x and y directions make sense as in the figures above.

- Closed Subsets in the plane: a bounded set that contains all of its boundary points (the analogy of a closed interval in the line)

- Extreme Value Theorem: If $f : \mathbb{R}^2 \to \mathbb{R}$ is continous and D is a closed subset of the plane, then f attains both an absolute minimum and absolute maximum value at points inside D.

- How to find Absolute Min/Max Values on a closed set D:
  Break up D into two parts, I = inside part (open set) of D, B = boundary curve
  Step 1: find critical points in I=inside D
  Step 2: find the points where f has extreme values in B
  To do this: parametrize the boundary curve (in pieces if necessary) with $(x(t), y(t))$, then find the extra of the one-variable function $f(t) = f(x(t), y(t))$ using Calc 1 techniques.
  Step 3: Evaluate f at points from Steps 1 and 2 and select the largest and smallest values.

- How to find Extrema on a closed set using Lagrange Multipliers:
  Let $f(x, y, z)$ and $g(x, y, z)$ be functions with continuous partial derivatives.
  To find the extremum of $f(x, y, z)$ subject to the constraint $g(x, y, z) = c$, solve the equations:

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = c \end{cases}$$

for x, y, z, and λ. That is, we solve: $f_x = \lambda g_x$, $f_y = \lambda g_y$, $f_z = \lambda g_z$, and $g = c$.

Chapter 15: Multiple Integrals

Summary:

- dA=infinitesimal unit of area:
  - Cartesian Coordinates in the plane: $dA = dxdy$
  - Polar Coordinates in the plane: $dA = rdrd\theta$
- dV=infinitesimal unit of volume:
  - Cartesian Coordinates in space: $dV = dxdydz$
  - Cylindrical Coordinates in space: $dV = rdrd\theta dz$
  - Spherical Coordinates in space: $dV = \rho^2 \sin(\phi)\, d\rho d\theta d\phi$

More details:

- Definition of a double integral as a limit
- Double Integrals of functions $f(x, y)$ over rectangles $R = [a, b] \times [c, d]$ as iterated integrals
- Geometric Interpretation of $\iint_D f(x, y)\, dA$: Volume under the graph of the surface $z = f(x, y)$ (when $f(x, y) \geq 0$) lying above the rectangle R in the plane.
- Fubini's Theorem:
  When integrating over a rectangle, you can do the integrals in any order!
  $$\iint_R f(x, y)\, dA = \int_a^b \left[\int_c^d f(x, y)\, dy\right] dx = \int_c^d \left[\int_a^b f(x, y)\, dx\right] dy$$
- Area a domain D in the plane: $\text{Area}(D) = \iint_D 1\, dA$.
- Double Integrals over Elementary Domains D in the plane:
  - D is Type I:
  $$D : \begin{cases} a \leq x \leq b \\ g_1(x) \leq y \leq g_2(x) \end{cases} \implies \iint_D f dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\right] dx$$
  - D is Type II:
  $$D : \begin{cases} c \leq y \leq d \\ h_1(y) \leq x \leq h_2(y) \end{cases} \implies \iint_D f dA = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\right] dy$$
- FACT: if f is continuous on the elementary region D, then the double integral over D exists.
- Be able to compute double integrals of Type I or II fully. But also be able to set-up the correct integrals. Given an integral, be able to read and sketch the domain and switch the order of integration.
- Double Integrals in Polar Coordinates:
  Given cartesian coordinates (x, y), the equations for polar coordinates are: $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$.
  Given polar coordinates (r, θ), the equations for cartesian coordinates are: $x = r\cos(\theta)$ and $y = r\sin(\theta)$.
  The infinitesimal unit of area is: $\boxed{dA = r\, dr\, d\theta}$

• When D can be easily described by polar coordinates as a sector (circles, quarter circles, annuli, etc):

$$D : \begin{cases} a \le r \le b \\ \alpha \le \theta \le \beta \end{cases} \implies \iint_D f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos(\theta), r\sin(\theta))\, r dr\, d\theta$$

or $\int_a^b \int_\alpha^\beta f(r\cos(\theta), r\sin(\theta))\, r d\theta\, dr$ by Fubini's Theorem.

• When D is a more general region in PC:
When the "wobbly sector" i.e. $r = h_1(\theta)$ is a lower bound for r and $r = h_2(\theta)$ is an upper bound for r:

$$D : \begin{cases} \alpha \le \theta \le \beta \\ h_1(\theta) \le r \le h_2(\theta) \end{cases} \implies \iint_D f(x,y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos(\theta), r\sin(\theta))\, r dr\, d\theta$$

- Be able to find the area of regions described using PC

- Triple Integrals of $f(x,y,z)$ over boxes $B = [a,b] \times [c,d] \times [r,s]$ using iterated integrals

- Geometric Interpretation of $\iiint_E f(x,y,z)\, dV$: We can't visualize this! The units of this integral are 4-dimensional! It sums up the values of the function $f(x,y,z)$ times the infinitesimal volume dV as (x,y,z) ranges over the solid E in space.
  Best way to think of it: $T(x,y,z)$ is temperature at point (x,y,z) in the oven B then $\iiint_B T(x,y,z)\, dV$ is the total temperature inside B.

- Fubini's Theorem:
  When integrating over a box, you can do the integrals in any order!
  $$\iiint_B f(x,y,z)\, dV = \int_a^b \left[\int_c^d \left[\int_r^s f(x,y,z)\, dz\right] dy\right] dx = \int_a^b \left[\int_r^s \left[\int_c^d f(x,y,z)\, dy\right] dz\right] dx$$
  and equal to any of the other 4 possibilities.

- Volume of a region E in space: $\text{Vol}(E) = \iiint_E 1\, dV$.

- Triple Integrals over Elementary Regions E in space:
  • E is Type I:

$$E : \begin{cases} (x,y) \in D \\ u_1(x,y) \le z \le u_2(x,y) \end{cases} \implies \iiint_E f dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz\right] dA$$

then depending on whether D is Type I or Type II:

$$\iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz\right] dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz\right] dy\right] dx \qquad (D \text{ is Type I})$$

$$\iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz\right] dA = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz\right] dx\right] dy \qquad (D \text{ is Type II})$$

• E is Type II:

$$E : \begin{cases} (y,z) \in D \\ u_1(y,z) \le x \le u_2(y,z) \end{cases} \implies \iiint_E f dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\, dx\right] dA$$

then depending on whether D is Type I or Type II:

$$\iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right] dA = \int_c^d \left[\int_{g_1(y)}^{g_2(y)} \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right] dz\right] dy \qquad (D \text{ is Type I})$$

$$\iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right] dA = \int_r^s \left[\int_{h_1(z)}^{h_2(z)} \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right] dy\right] dz \qquad (D \text{ is Type II})$$

• E is Type III:

$$E : \begin{cases} (x,z) \in D \\ u_1(x,z) \le y \le u_2(x,z) \end{cases} \implies \iiint_E f dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right] dA$$

then depending on whether D is Type I or Type II:

$$\iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right] dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right] dz\right] dx \qquad (D \text{ is Type I})$$

$$\iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right] dA = \int_r^s \left[\int_{h_1(z)}^{h_2(z)} \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right] dx\right] dz \qquad (D \text{ is Type II})$$

- Important examples are to compute the volume of spheres using either Type I, II, or III triple integrals.

- Triple Integrals in Cylindrical Coordinates:
  Cylindrical coordinates: (r,θ,z)
  Given cartesian coordinates (x,y,z), the equations for cylindrical coordinates are: $x^2+y^2=r^2$, $\theta=\tan^{-1}(y/x)$, and $z=z$.
  Given cylindrical coordinates (r,θ,z), the equations for cartesian coordinates are: $x=r\cos(\theta)$, $y=r\sin(\theta)$, and $z=z$.
  The infinitesimal unit of volume is: $\boxed{dV = r\,dr\,d\theta\,dz}$
  • When E can be easily described by cylindrical coordinates as a cylinder (or part of):

$$E : \begin{cases} a \le r \le b \\ \alpha \le \theta \le \beta \\ r \le z \le s \end{cases} \implies \iiint_E f(x,y,z)dV = \int_r^s \int_\alpha^\beta \int_a^b f(r\cos(\theta), r\sin(\theta), z)\, r dr\, d\theta\, dz$$

  or in any of the other 5 possible orders of $dr, d\theta, dz$ by Fubini's Theorem.
  • When E is a more general region in CC:
  Besides cylinders know the equation of cone in CC: $z=r$. So you can describe regions like an "ice cream cone"

- Triple Integrals in Spherical Coordinates:
  Spherical coordinates: (ρ,θ,ϕ)
  Given cartesian coordinates (x,y,z), the equations for Spherical coordinates are: $\rho^2=x^2+y^2+z^2$, $\theta=\tan^{-1}(y/x)$, and and $\phi=\cos^{-1}(z/\rho)$.
  Given Spherical coordinates (ρ,θ,ϕ), the equations for cartesian coordinates are: $x=(\rho\sin(\phi))\cos(\theta)$, $y=(\rho\sin(\phi))\sin(\theta)$, and $z=\rho\cos(\phi)$.
  The infinitesimal unit of volume is: $\boxed{dV = \rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi}$
  • When E can be easily described by Spherical coordinates as a sphere (or part of):

$$E: \begin{cases} a \le \rho \le b \\ \alpha \le \theta \le \beta \\ \delta \le \phi \le \gamma \end{cases} \implies$$

$$\iiint_E f(x,y,z)dV = \int_\delta^\gamma \int_\alpha^\beta \int_a^b f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\, \rho^2 \sin(\phi) d\rho\, d\theta\, d\phi$$

or in any of the other 5 possible orders of $d\rho, d\theta, d\phi$ by Fubini's Theorem.

• When E is a more general region in SC:

Besides spheres know the equation of cone in CC: ϕ =constant. So you can describe regions like an "ice cream cone"

Chapter 16: Vector Calculus

- Vector Fields: a vector field $\vec{F}$ gives a vector (in plane or in space) at every point.
  More generally, vector fields are functions: $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$
  • VFs in the Plane: $\boxed{\vec{F} = \langle P, Q\rangle}$
  $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$, $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ where $P, Q: \mathbb{R}^2 \to \mathbb{R}$ are SVFs.
  • VFs in Space: $\boxed{\vec{F} = \langle P, Q, R\rangle}$
  $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$, $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ where $P, Q, R: \mathbb{R}^3 \to \mathbb{R}$ are SVFs.

- Visualization of a vector field as a "field of arrows" and interpretation as a force field, or fluid flow

- Important examples: (a) "Explosion" $\vec{F}(x,y) = \langle x, y\rangle$; (b) "Implosion" $\vec{F}(x,y) = -\langle x, y\rangle$; (c) "Circulation" counter-clockwise $\vec{F}(x,y) = \langle -y, x\rangle$; (c) "Circulation" clockwise $\vec{F}(x,y) = \langle y, -x\rangle$

- Gradient Vector Fields: $\boxed{\nabla f = \langle f_x, f_y, f_z\rangle}$

- Recall: curves in the plane and in space:
  $\boxed{\vec{r}(t) = \langle x(t), y(t), z(t)\rangle}$ and $\boxed{ds = |\vec{r}'(t)|\, dt}$
  since $ds = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt = |\vec{r}'(t)|dt$.
  Infinitesimal unit of vector arclength: $d\vec{r} = \vec{T}(t)ds$.
  But this is a pain to compute, so instead we use: $\boxed{d\vec{r} = \vec{r}'(t)\, dt}$

- LINE INTEGRAL OF $\vec{F}$ ALONG A CURVE C: $\int_C \vec{F} \cdot d\vec{r}$.
  General: $\boxed{\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)dt}$ (Notice: this uses the DOT product!)

  In the plane: $\boxed{\int_C \langle P, Q\rangle \cdot d\vec{r} = \int_C Pdx + Qdy}$

  Notice: $\vec{F} = \langle P, Q\rangle$ and $d\vec{r} = \vec{r}'(t)dt = \langle x'(t), y'(t)\rangle dt$, so computing the dot product gives: $\vec{F} \cdot d\vec{r} = \langle P, Q\rangle \cdot \langle x'(t), y'(t)\rangle dt = Px'(t)dt + Qy'(t)dt = Pdx + Qdy$
  since $\boxed{dx = x'(t)dt}$ and $\boxed{dy = y'(t)dt}$

  In space: $\boxed{\int_C \langle P, Q, R\rangle \cdot d\vec{r} = \int_C Pdx + Qdy + Rdz}$

- Geometric Meaning of a line integral of a vector field along a closed curve C: Circulation of $\vec{F}$ along the curve C

- Know how to parametrize curves: line segments, circles, ellipses, parabolas, squares, triangles, etc

- Properties of curves: orientation, $C_1 \cup C_2$, $-C$ etc

- Properties of Line integrals: $\int_{C_1 \cup C_2} \vec{F} = \int_{C_1} \vec{F} + \int_{C_2} \vec{F}$ and $\int_{-C} \vec{F} = -\int_C \vec{F}$.

- DEFINITIONS/TERMINOLOGY:
  Definition of $\vec{F}$ path independent
  Curves C: Closed, Simple
  Domains D: Open, connected, simply connected
  NOTATION: $\boxed{\partial D = C}$ is the notation for the boundary curve of D. It comes with orientation defined by: positive when traveling along the boundary curve, the domain D is on your left side. Negative when traveling along the boundary curve, the domain D is on your right side.

- CONSERVATIVE VECTOR FIELDS
  Definition of $\vec{F}$ conservative
  THM $\vec{F}$ conservative $\iff \oint_C \vec{F} = 0$ for all closed loops
  THM $\vec{F}$ conservative $\iff$ it is the gradient of some function, ie $\vec{F} = \nabla f$
  Note: f is called a Potential function. Know how to find f if given a conservative VF
  THM (Fundamental Thm of Line Integrals): $\int_C \nabla f(\vec{r}) \cdot d\vec{r} = f(B) - f(A)$
  (where C a curve from A to B)
  THM (Fundamental Theorem of Conservative VFs):
  Let D be a simply connected domain in the plane. Then
  $\vec{F} = \langle P, Q \rangle$ is conservative on $D \iff \boxed{\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}}$ on D

- GREEN'S THEOREM
  Assumptions needed:
  • D simply connected domain in the plane (=open+connected+no holes or punctures)
  • $\partial D = C$ the boundary curve is a simple, closed curve oriented positive sense (ie CCW)
  • $\vec{F} = \langle P, Q \rangle$ with P, Q continuous partial derivatives inside D and on ∂D

  $$\text{THEN } \boxed{\oint_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA}$$

  WARNING: $\vec{F}$ must be defined and differentiable inside D for you to apply Green's Theorem

- Scalar Curl: S.Curl$(\vec{F}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$
  Meaning: the infinitesimal circulation of $\vec{F}$ at the point (x, y)

- Vector Form of Green's Theorem: $\boxed{\oint_{\partial D} \vec{F}(\vec{r}) \cdot d\vec{r} = \iint_D \text{S.Curl}(\vec{F})\, dA}$

BONUS: GRADIENT OPERATOR, CURL, & DIVERGENCE

- Del Operators: $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle$ in 2D and $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$ in 3D

- CURL of F: $\boxed{\text{Curl}(\vec{F}) = \nabla \times \vec{F}}$ only for 3D $\vec{F} = \langle P, Q, R \rangle$

  $$\text{Curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

  NOTE: Curl$(\vec{F})$ is clearly a vector!
  Geometric Meaning: the circulation at a point through a plane orthogonal to Curl$(\vec{F})$

- DIVERGENCE of F: $\boxed{\text{div}(\vec{F}) = \nabla \cdot \vec{F}}$
  $\text{div}(\vec{F}) = \nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle \cdot \langle P, Q \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$.
  $\text{div}(\vec{F}) = \nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle P, Q \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.
  Geometric Meaning: the contribution of $\vec{F}$ in the direction of the "explosion vector field" at a point. This is termed "flux" of the vector field.

BONUS: INTEGRATION OVER SURFACES

- Recall Surfaces in space
  you can define a surface via a function $f : \mathbb{R}^2 \to \mathbb{R}$ with $z = f(x, y)$
  you can define a surface implicitly via a function $f : \mathbb{R}^3 \to \mathbb{R}$ with $f(x, y, z) = c$
- For simplicity, we only study integrals over surfaces defined as $z = f(x, y)$ over a domain D in the plane. The domain D is the range of values for x and y (think back to double and triple integrals from previous chapters)
- Given a surface $S : z = f(x, y)$
  Infinitesimal piece of surface area: $dA = \sqrt{1 + (f_x)^2 + (f_y)^2}\, dxdy$
  Normal vector to S at a point: $\vec{n} = \langle -f_x, -f_y, 1 \rangle$ (outward pointing)
  Recall this comes from: $\vec{n} = \vec{f_x} \times \vec{f_y} = \langle 1, 0, f_x \rangle \times \langle 0, 1, f_y \rangle$
  Unit Normal: $\hat{n} = \dfrac{\vec{n}}{\|\vec{n}\|} = \dfrac{\langle -f_x, -f_y, 1 \rangle}{\sqrt{1 + (f_x)^2 + (f_y)^2}}$
  Oriented infinitesimal area: $d\vec{A} = \hat{n} dA = \frac{\langle -f_x, -f_y, 1 \rangle}{\sqrt{1+(f_x)^2+(f_y)^2}} dA = \vec{n} dxdy$ so $\boxed{d\vec{A} = \vec{n} dxdy}$
  OR $\boxed{d\vec{A} = \langle -f_x, -f_y, 1 \rangle\, dxdy}$
- SURFACE INTEGRAL OF $\vec{\Phi}$ THROUGH S: $\iint_S \vec{F} \cdot d\vec{A}$.
  General: $\boxed{\iint_S \vec{F} \cdot d\vec{A} = \iint_D \vec{F}(x, y) \cdot \vec{n}\, dxdy}$
  $\boxed{\iint_S \vec{F} \cdot d\vec{A} = \iint_D \vec{F}(x, y) \cdot \langle -f_x, -f_y, 1 \rangle\, dxdy}$
  Alternate Form: $\boxed{\iint_S \langle P, Q, R \rangle \cdot d\vec{A} = \iint_D -P f_x\, dx - Q f_y\, dy + R\, dz}$
  Geometric Meaning: "Flux" of $\vec{F}$ across/through the surface S

BONUS: STOKE'S THEOREM

- STOKE'S THEOREM
  Assumptions needed:
  • D and ∂D are planar domain and boundary curve that satisfy assumptions of Green's Theorem
  • S and ∂S is a surface in space of the form $z = f(x, y)$ over the domain D and $f(\partial D) = \partial S$ (this just says that the function f evaluated over the boundary curve in the plane gives the boundary curve ∂S of the surface S in space)
  • orientation ∂S is oriented in the positive sense (the surface is always on your left as you walk around the boundary)
  • orientation S is oriented in the positive sense (outward pointing normal vector)
  THEN $\boxed{\oint_{\partial S} \vec{F}(\vec{r}) \cdot d\vec{r} = \iint_S \text{Curl}(\vec{F}) \cdot d\vec{A}}$

Equivalently: $\boxed{\oint_{\partial S} \vec{\Phi}(\vec{r}) \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{A}}$

Or: $\boxed{\oint_{\partial S} Pdx + Qdy + Rdz = \iint_S -f_x(R_y - Q_z) - f_y,(P_z - R_x) + (Q_x - P_y)\,dxdy}$

BONUS: FLUX and DIVERGENCE

- FLUX of $\vec{F}$ ACCROSS C: $\int_C \vec{\Phi} \cdot \hat{n} ds$.
  Geometric meaning: the contribution of $\vec{F}$ across the curve C
- Formula for $\hat{n}ds$:
  • parametrize C with $\vec{r}(t) = \langle x(t), y(t) \rangle$
  • ds=infinitesimal piece of arclength of the curve C: $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$
  • $\vec{n}$ = normal vector: outward pointing vector that is orthogonal to the tangent vector $\vec{r}'(t)$
  • $\boxed{\vec{n} = \langle \frac{dy}{dt}, -\frac{dx}{dt} \rangle}$
  • $\hat{n}$ = unit normal vector: $\hat{n} = \frac{\vec{n}}{\|\vec{n}\|} = \frac{\langle \frac{dy}{dt}, -\frac{dx}{dt} \rangle}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}$
  • All of these simply to: $\boxed{\hat{n}ds = \langle \frac{dy}{dt}, -\frac{dx}{dt} \rangle dt}$
- Alternate form of flux using $F(x,y) = \langle P, Q \rangle$: $\int_C \vec{F} \cdot \hat{n} ds = \int_C -Qdx + Pdy$.
- GAUSS' DIVERGENCE THEOREM in the plane: $\boxed{\int_C \vec{F} \cdot \hat{n} ds = \iint_D (\nabla \cdot \vec{F})\, dxdy}$
- GAUSS' DIVERGENCE THEOREM in space: $\boxed{\iint_{\partial E} \vec{\Phi} \cdot d\vec{A} = \iiint_E (\nabla \cdot \vec{\Phi})\, dxdydz}$
  where E is a solid region in space and ∂E is the surface which is the boundary of E
  Note: $\text{div}(\vec{F}) = \nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

PRACTICE EXAMS

PRACTICE PROBLEMS FOR EXAM 1

Chapter 12 Review Problems: pages 841-843

- Concept Check: 1-19 all
- True-False Quiz: 1-22 all
- Exercises: 1, 3–7, 11, 12, 15–21, 26–34

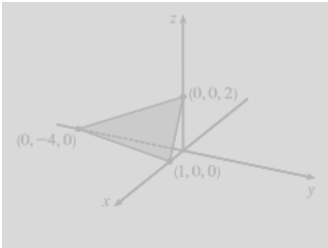
Chapter 13 Review Problems: pages 881-883

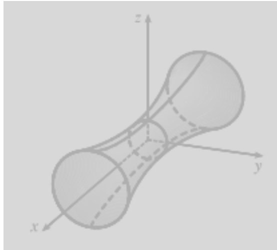
- Concept Check: 1-5 all
- True-False Quiz: 1–6 all, 11, 12
- Exercises: 1, 2, 3, 5

Answers: Ch 12 R
CC will be uploaded separately (see Sakai)
T/F: FFFFT FTTTT TFTTF FFFFF TT
Ex For odd answers see back of book.

Even: 4. (a) $11\hat{\imath}-4\hat{\jmath}-\hat{k}$ (b) $\sqrt{14}$ (c) -1 (d) $-3\hat{\imath}-7\hat{\jmath}-5\hat{k}$ (e) $9\hat{\imath}+15\hat{\jmath}+3\hat{k}, 3\sqrt{35}$ (f) 18 (g) $\vec{0}$ (h) $33\hat{\imath}-21\hat{\jmath}+6\hat{k}$ (i) $-1/\sqrt{6}$ (j) $(-1/6)(\hat{\imath}+\hat{\jmath}-2\hat{k})$ (k) $\cos^{-1}(-1/(2\sqrt{21}))$. 6. $\pm(7\hat{\imath}+2\hat{\jmath}-\hat{k})/(3\sqrt{6})$. 12. $\vec{D}=\langle 4,3,6\rangle$ and $W=87J$. 16. $x=1+3t, y=2t, z=-1+t$. 18. $(x-2)+4(y-1)-3(z-0)=0$. 20. $6x+9y-z=26$. 26. (a) $x+3y+z=6$ (b) $\frac{x+1}{1}=\frac{y+1}{3}=\frac{z-10}{1}$ (c) $\cos^{-1}(-13/\sqrt{319})\approx 137°$ so $180°-137°=43°$ (d) $x=2+t, y=-t, z=4+2t$, 28. plane parallel to yz-plane passing through $(3,0,0)$.

30. 

32. 

34. 

Answers: Ch 13 R
CC will be uploaded separately (see Sakai)
T/F: TTFTF FFFTT FTTF
Ex For odd answers see back of book.
Even: 2. $D=(-1,0)\cup(0,2]$

PRACTICE PROBLEMS FOR EXAM 2

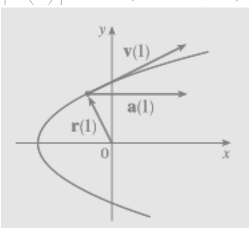
Chapter 13 Review Problems: pages 881-883

- Concept Check: 6,7,8
- True-False Quiz: 8,10
- Exercises: 6,8,9,11,17-19

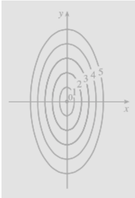
Chapter 14 Review Problems: pages 981-984

- Concept Check: 1-18 all
- True-False Quiz: 1–12 all
- Exercises: 1-6, 8-10, 11a,b, 12-17, 19-29, 32-38, 42-48, 51-56, 63

Answers: Ch 13 R
CC will be uploaded separately (see Sakai)
T/F: F T
Ex For odd answers see back of book.
Even: 6. (a) $(15/8,0,-\ln(2))$ (b) $x=1-3t, y=1+2t, z=t$ (c) $-3(x-1)+2(y-1)+z=0$. 8. $L=\int_0^1\sqrt{9t+4}dt=\int_4^{13}\sqrt{u}du/9=(2/27)(13^{3/2}-8)$. 18. velocity = $\vec{v}(t)=\langle 4t,2\rangle$, speed = $|\vec{v}(t)|=2\sqrt{4t^2+1}$, acceleration = $\vec{a}(t)=\langle 4,0\rangle$. At $t=1$, $\vec{r}(1)=\langle -1,2\rangle$, $\vec{v}(1)=\langle 4,2\rangle$, $\vec{a}(1)=\langle 4,0\rangle$.



Answers: Ch 14 R
CC will be uploaded separately (see Sakai)
T/F: TFFTF FTFFT TF
Ex For odd answers see back of book.
Even: 2. $\{(x,y)\mid -1\le x\le 1, -\sqrt{4-x^2}\le y\le\sqrt{4-x^2}\}$. 4. A circular paraboloid opening up centered

at $(0,2,0)$. 6. 

8. (a) $f(3,2) \approx 55$ (b) $f_x(3,2) < 0$ since if we fix y at $y = 2$ and allow x to vary, the level curves indicate that the z-values decrease as x increases. (c) Both $f_y(2,1)$ and $f_y(2,2)$ are positive, because if we start from either point and move in the positive y-direction, the contour map indicates that the path is ascending. But the level curves are closer together in the y-direction at $(2,1)$ than at $(2,2)$, so the path is steeper (the z-values increase more rapidly) at $(2,1)$ and hence $f_y(2,1) > f_y(2,2)$. 10. DNE. Choose two paths. Limit is 0 along the x-axis, but limit is 2/3 along the line $y = x$. 12. Linearization of T at $(6,4)$ is $L(x,y) = 80 + 3.5(x-6) - 3.0(y-4)$. So, $T(5,3.8) \approx L(5,3.8) = 77.1°C$. 14. $g_u(u,v) = \frac{v^2-u^2-4uv}{(u^2+v^2)^2}$, $g_v(u,v) = \frac{2u^2-2v^2-2uv}{(u^2+v^2)^2}$. 16. $G_x(x,y,z) = ze^{xz}\sin(y/z)$, $G_y(x,y,z) = (e^{xz}/z)\cos(y/z)$, $G_z(x,y,z) = e^{xz}[x\sin(y/z) - (y/z^2)\cos(y/z)]$. 20. $z_{xx} = 0$, $z_{xy} = z_{yx} = -2e^{-y}$, $z_{yy} = 4xe^{-y}$. 22. $\frac{\partial^2 v}{\partial r^2} = 0$, $\frac{\partial^2 v}{\partial s^2} = -r\cos(s+2t)$, $\frac{\partial^2 v}{\partial t^2} = -4r\cos(s+2t)$, $\frac{\partial^2 v}{\partial r\partial s} = \frac{\partial^2 v}{\partial s\partial r} = -\sin(r+2t)$, $\frac{\partial^2 v}{\partial r\partial t} = \frac{\partial^2 v}{\partial t\partial r} = -2\sin(r+2t)$, $\frac{\partial^2 v}{\partial t\partial s} = \frac{\partial^2 v}{\partial s\partial t} = -2r\cos(r+2t)$. 24. True. 26. (a) $z = x+1$ (b) $x = t, y = 0, z = 1-t$. 28. (a) $2(x-1)+2(y-1)+2(z-1) = 0$ (b) $x = t, y = t, z = t$. 32. $du = u_s ds + u_t dt = (\frac{e^{2t}}{1+se^{2t}})ds + (\frac{2se^{2t}}{1+se^{2t}})dt$ 34. (a) $A = \frac{1}{2}bh$, $dA = A_b db + A_h dh = (\frac{h}{2})db + (\frac{b}{2})dh$, max error is $dA = (2.5)(0.002) + (6)(0.002) = 0.017m^2$ (notice: conver to same units!) (b) $c = \sqrt{b^2+h^2}$, $dc = c_b db + c_h dh = (\frac{b}{\sqrt{b^2+h^2}})db + (\frac{h}{\sqrt{b^2+h^2}})dh$, max error is $dc = (5/13)(0.002) + (12/13)(0.002) = 0.0026m$. 36. $\frac{\partial v}{\partial s}|_{(0,1)} = 5$, $\frac{\partial v}{\partial t}|_{(0,1)} = 0$. 38.

$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial p} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial p} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial p}$, $\frac{\partial w}{\partial q} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial q} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial q} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial q}$, $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial r} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial r}$, $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial s} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial s}$. 42. $\frac{\partial z}{\partial x} = \frac{-2xy^2 - yz\sin(xyz)}{2z + xy\sin(xyz)}$, $\frac{\partial z}{\partial y} = \frac{-2x^2 - xz\sin(xyz)}{2z + xy\sin(xyz)}$. 44. (a) By Theorem 14.6.15, the max value of the directional derivative occurs when $\vec{u}$ has the same direction as the gradient vector. (b) It is a minimum when $\vec{u}$ is in the direction opposite to that of the gradient vector (that is, $\vec{u}$ is in the direction of $-\nabla f$), since $D_{\vec{u}}(f) = |\nabla f|\cos\theta$ has a minimum at $\theta = \pi$. (c) The directional derivative is zero when $\vec{u}$ is perpendicular to the gradient vector since then $D_{\vec{u}}(f) = \nabla f \cdot \vec{u} = 0$ (d) The directional derivative is half of its maximum value when $D_{\vec{u}}(f) = |\nabla f|\cos\theta = \frac{1}{2}|\nabla f|$ when $\theta = \pi/3$. 46. $D_{\vec{u}}(f)(1,2,3) = 25/6$. 48. $\nabla f(0,1,2) = \langle 2,0,1\rangle$, max rate is $|\nabla f(0,1,2)| = \sqrt{5}$. 52. CP: $(0,0)$, $(1,1/2)$, $(0,0)$ is a saddle point, $(1,1/2)$ is a local min, so $f(1,1/2) = -1$ is a local min. 54. CP: $(0,-2)$ only, $(0,2)$ is a local min, so $f(0,-2) = -2/e$ is a local min. 56. The absolute max is $2/e$ and it occurs at $(0,\pm 1)$, the absolute min is 0 and it occurs at $(0,0)$.

PRACTICE PROBLEMS FOR EXAM 3

Chapter 14 Review Problems: pages 981-984

- Concept Check: 19
- True-False Quiz: N/A
- Exercises: 59, 61

Chapter 15 Review Problems: pages 1061-1064

- Concept Check: 1a-d, 2b-d, 3, 7, 9
- True-False Quiz: 1-7, 9
- Exercises: 3, 5, 7, 9, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35–38, 40, 47, 48, 53

Answers: Ch 14 R
CC will be uploaded separately (see Sakai)
T/F: F T

Ex For odd answers see back of book.
Even: N/A
Answers: Ch 15 R
CC will be uploaded separately (see Sakai)
T/F: TFTTT TTF
Ex For odd answers see back of book.
Even: 36. Type I Triple Integral: D is type II in the xy-plane: $0 \leq y \leq 1$, $y+1 \leq x \leq 4-2y$, and $0 \leq z \leq x^2y$. Thus, $V = \int_0^1 \int_{y+1}^{4-2y} \int_0^{x^2y} 1\, dz\, dx\, dy = 53/20$. 38. Use Cylindrical Coordinates: E: $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 2$, and $0 \leq z \leq 3-y = 3 - r\sin(\theta)$. Thus, $V = \int_0^{2\pi} \int_0^2 \int_0^{3-r\sin(\theta)} r\, dz\, dr\, d\theta = 12\pi$. 40. The paraboloid and half-cone intersect when $x^2+y^2 = \sqrt{x^2+y^2}$ or when $x^2+y^2 = 1$ or $x^2+y^2 = 0$. So, $D = \{(x,y) \mid x^2+y^2 \leq 1\}$. So, $V = \iint_D \left[\int_{x^2+y^2}^{\sqrt{x^2+y^2}} 1\, dz\right] dA = \int_0^{2\pi} \int_0^1 \int_{r^2}^{r} r\, dz\, dr\, d\theta = \pi/6$. 48. E is the solid hemisphere $x^2+y^2+z^2 \leq 4$ with $x \geq 0$. In Spherical coordinates: $0 \leq \rho \leq 2$, $-\pi/2 \leq \theta \leq \pi/2$, $0 \leq \phi \leq \pi$. We change $y^2\sqrt{x^2+y^2+z^2}$ into spherical coordinates $(\rho\sin(\phi)\sin(\theta))^2(\sqrt{\rho^2})^2 = \rho^3\sin^2(\phi)\sin^2(\theta)$. So the integral becomes: $\int_0^{\pi} \int_{-\pi/2}^{\pi/2} \int_0^2 (\rho^3\sin^2(\phi)\sin^2(\theta))(\rho^2\sin(\phi))\, d\rho\, d\theta\, d\phi = 64\pi/9$.

PRACTICE PROBLEMS FOR FINAL EXAM

Chapter 16 Review Problems: pages 1148-1150

Note: Starred problems are optional and may show up only as extra credit problems.

- Concept Check: 1, 2, 3a,b, 4, 5, 6, 7, 9*, 10, 14*
- True-False Quiz: 1*, 2*, 4, 5, 6
- Exercises: 1a, (b*), 3-15 odd, 16, 17, 18*, 29*, 31*

Answers: Ch 16 R
CC will be uploaded separately (see Sakai)
T/F: 1. T. F. 2. T. 4. T. 5. F. 6. F.
Ex For odd answers see back of book.
Even: 16. $\int_C \sqrt{1+x^3}dx + 2xydy = \iint_D [2y - 0]\, dA = \int_0^1 \int_0^{3x} (2y)\, dy\, dx = 3$. 18. $\text{Curl}(\vec{F}) = \nabla \times \vec{F} = \langle (0 - e^{-y}\cos(z)), -(e^{-z}\cos(x) - 0), (0 - e^{-x}\cos(y)) \rangle = \langle -e^{-y}\cos(z), -e^{-z}\cos(x), -e^{-x}\cos(y) \rangle$, $\text{div}(\vec{F}) = \nabla \cdot \vec{F} = -e^{-x}\sin(y) - e^{-y}\sin(z) - e^{-z}\sin(x)$.