SARAH · LAWRENCE · COLLEGE

HHW# 15 - Final FALL 2016

Math 3614. Complex Analysis. Jorge Eduardo Basilio

Posted on Monday morning.

DIRECTIONS

- All work turned in must be NEAT, ORGANIZED, AND STAPLED. Handwriting should be neat and legible.
- Also, it must have a cover page with the following information: the Course Name, Section, Instructor's name (me!), Date Due, name of the assignment and your name all neatly and clearly marked. In addition, write the following statement and sign-your name under it: *"I swear to neither receive nor give any unauthorized assistance on this assignment."*
- It must be written on blank or lined 8.5" by 11" sized paper using only the front side of the page.
- It can be written in pen or pencil. If written in pen, mistakes must be neatly crossed out, or erased with white-out.
- Be sure to label each problem carefully and staple work sequentially.
- Be sure to show all work on every problem.
- The level of "polish" should be very high, just like it would be when turning in an essay in an English class. You should work out solutions on scratch paper before you write your final draft. When applicable, write in complete sentences.
- This assignment is open book and open notes (from seminar, RPs, or previous HHWs), however, consulting other textbooks or the internet is strictly forbidden.
- You *may* work with other classmates but writing up solutions to the problems must be done entirely on your own (i.e. no copying other peoples written solutions but working together on the blackboard is fine!). Please acknowledge collaborations on each individual problem (if applicable).

DUE TUESDAY, DECEMBER 20, 5:00 PM

Due online on your Google drive as a single PDF file.

Problem 1. $\mathbb{C}$.

Fix any $a \in \mathbb{C}$. For z such that $\bar{a}z \neq 1$, define

$$A(z) = \frac{z-a}{1-\bar{a}z}.$$

Show that $|A(z)| = 1$ if z is any point on the unit circle (i.e. $|z| = 1$).

Problem 2. Visual mappings.

Let $S = [0, \pi/3] \times [0, 1]$ is the rectangle $\{x + iy : 0 \leq x \leq \pi/3, 0 \leq y \leq 1\}$, draw the image, $S' = f(S)$, of S under the mapping $f(z) = e^{iz}$.

Problem 3. $\mathbb{C}$ as vectors in the place.

Let $f(z) = z + \frac{1}{z}$. Demonstrate that the image of the unit circle $C = \{z \in \mathbb{C} : |z| = 1\}$ is the real line interval $[-2, 2] \subset \mathbb{C}$

(a) using the vector representation of complex numbers;

(b) using the polar representation of complex numbers.

Problem 4. $\mathbb{C}$ Integral I.

Evaluate

$$\oint_\gamma \left(z + \frac{1}{z - \pi/2}\right)^3 dz,$$

where γ is the unit circle oriented in the positive sense (counter-clockwise).

Problem 5. $\mathbb{C}$ Integral II.

Use the Cauchy Integral Formulas to evaluate

$$\oint_\gamma \frac{\sin^3(z)}{(z - \pi/6)^3}\, dz,$$

where γ is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ oriented in the positive sense (counter-clockwise).

Problem 6. Disks of Convergence.

Find the center and radius of convergence for the following series:

(a) $\displaystyle\sum_{k=5}^{\infty} \frac{3^k}{k} z^k$ (b) $\displaystyle\sum_{k=0}^{\infty} \frac{2^k}{k!} z^{3k}$ (c) $\displaystyle\sum_{k=0}^{\infty} k! z^{k!}$ (d) $\displaystyle\sum_{k=1}^{\infty} \frac{(z+2)^{k-1}}{(k+1)4^k}$

Problem 7. Laurent Series Representation.

Find:
(a) the Laurent Series Representation of

$$f(z) = z^5 e^{1/z^2}$$

at $z_0 = 0$;

(b) the largest annulus of convergence for the LSR found in (a);

(c) and classify the singularity at $z_0 = 0$.

Problem 8. Removable Singularities.

Fix a disk $D = D(z_0, R)$. Let f be analytic on the punctured disk $D^\times = D \setminus \{z_0\}$. So, z_0 is an isolated singularity of f. Let $f(z) = \sum_{-\infty}^{\infty} a_k (z - z_0)^k$ be it's Laurent Series Representation on D where, by Laurent's Theorem,

$$a_k = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{k+1}}\, dz. \qquad (*)$$

Recall that we defined f to have a *removable singularity at* z_0 on the punctured disk $D^\times = D \setminus \{z_0\}$ if $a_k = 0$ for all $k < 0$. In this problem you'll show:

Riemann's Theorem on Removable Singularities:
f has a removable singularity at z_0 if and only if $\lim_{z \to z_0} f(z)$ exists and f is bounded on D.

You'll prove this result in the following steps:

(a) Show the ($\Longrightarrow$) direction, i.e. if f has a removable singularity at z_0 then $\lim_{z \to z_0} f(z)$ exists and f is bounded on D.

(b) Show the ($\Longleftarrow$) direction, i.e if $\lim_{z \to z_0} f(z)$ exists and f is bounded on D, then f has a removable singularity at z_0. [Hint: use $(*)$]

Problem 9. Characterization of Poles.

Fix a disk $D = D(z_0, R)$. Let f be analytic on the punctured disk $D^\times = D \setminus \{z_0\}$. So, z_0 is an isolated singularity of f. Let $f(z) = \sum_{-\infty}^{\infty} a_k (z - z_0)^k$ be it's Laurent Series Representation on D where, by Laurent's Theorem,

$$a_k = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{k+1}}\, dz. \qquad (*)$$

Recall that we defined f to have a *pole of order* $N > 0$ *at* z_0 on the punctured disk $D^\times = D \setminus \{z_0\}$ if $a_{-N} \neq 0$ but $a_k = 0$ for all $k < -N$. In this problem you'll show:

Characterization of Poles:
f has a pole of order N at z_0 if and only if $\lim_{z \to z_0} |f(z)| = +\infty$.

You'll prove this result in the following steps:

(a) Show the ($\Longrightarrow$) direction, i.e. if f has a pole of order N at z_0 then $\lim_{z \to z_0} |f(z)| = +\infty$. [Hint: Use Theorem 6.4.2]

(b) Show the ($\Longleftarrow$) direction, i.e if $\lim_{z \to z_0} |f(z)| = +\infty$, then f has a pole of order N at z_0. [Hint: use Riemann's Theorem of Removable Singularities with $g(z) = 1/f(z)$ and Theorem 6.4.3]

Problem 10. Classic Residue Calculus.

From elementary calculus, we know that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{R\to\infty}\int_{-R}^{R}\frac{1}{1+x^2}\,dx = \lim_{R\to\infty}\left[\tan^{-1}(x)\right]_{-R}^{R} = \pi.$$

Use contour integration and the method of residues to evaluate this integral and show that you get the same result.

Problem 11. Real Integral I.

Evaluate the real integral

$$\int_0^{2\pi} \frac{1}{1+3\cos(\theta)}\,d\theta$$

using the Residue Calculus, i.e. using the residue theorem.

Problem 12. Real Integral II.

Show that the principal value of the integral of $\sin(x)/x$ on the positive real axis is $\pi/2$, that is,

$$\int_0^{\infty} \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}.$$

HAVE A NICE HOLIDAY BREAK!