SARAH • LAWRENCE • COLLEGE

HHW# 8 - Midterm #1 FALL 2016

Math 3614. Complex Analysis. Jorge Eduardo Basilio

Posted on Tuesday Afternoon.

DIRECTIONS

The minimum grade one this assignment is a "Pass."
● To earn a "Pass"(ing) score on this assignment, answer CORRECTLY (roughly) 80% of the questions from "Warm ups! Parts 1 and 2."
● To earn a "C," you must earn a "Pass"(ing) score on the "Warm-Ups" plus solve Problems 1 and 2, plus ONE additional problem of your choosing from Problems 3-10.
● To earn a "B" you must earn a "Pass"(ing) score on the "Warm-Ups" plus solve Problems 3,4,5,6 plus ONE additional problem of your choosing from Problems 7-10.
● To earn an "A" you must earn a "Pass"(ing) score on the "Warm-Ups" plus solve Problems 5,6,7,8 plus ONE additional problem of your choosing from Problems 9 or 10.

- All work turned in must be NEAT, ORGANIZED, AND STAPLED. Handwriting should be neat and legible.
- Also, it must have a cover page with the following information: the Course Name, Section, Instructor's name (me!), Date Due, name of the assignment and your name all neatly and clearly marked. In addition, write the following statement and sign-your name under it: *"I swear to neither receive nor give any unauthorized assistance on this assignment."*
- It must be written on blank or lined 8.5" by 11" sized paper using only the front side of the page.
- It can be written in pen or pencil. If written in pen, mistakes must be neatly crossed out, or erased with white-out.
- Be sure to label each problem carefully and staple work sequentially.
- Be sure to show all work on every problem.
- The level of "polish" should be very high, just like it would be when turning in an essay in an English class. You should work out solutions on scratch paper before you write your final draft. When applicable, write in complete sentences.
- This assignment is open book and open notes (from seminar, RPs, or previous HHWs), however, you are to work entirely on your own (or you may ask me questions). Consulting other textbooks or the internet is strictly forbidden.

DUE THURSDAY, OCTOBER 20, 8:59 PM

You can slide it under my door at any time before the above deadline. Unless for extra-ordinary reasons, no late assignments will be counted/graded.

Warm ups! Part 1: True or False?

Please spell out in the entire word your answer (please don't write T or F only). If the statement is FALSE, then please write a short reason why it is false or simply give a counter-example.

__________ (i) $\text{Arg}(z)$ maps $\mathbb{C}$ into a subset of $\mathbb{R}$.

__________ (ii) $A = \{z \in \mathbb{C} : 1 \leq |z| < 3\}$ is both open and closed.

__________ (iii) i^{π} is represented by an infinite number of points lying somewhere on the unit circle, $\{z \in \mathbb{C} : |z| = 1\}$.

__________ (iv) e^z is a periodic function with Period $P = 2\pi$.

__________ (v) $\text{Ln}(\frac{1}{z}) = -\text{Ln}(z)$ for all $z \in \mathbb{C}$.

__________ (vi) There are complex numbers z such that $|\sin(z)| > 1$.

__________ (vii) If Im $(z) > 0$ then Re $(z) > 0$.

__________ (viii) The set $A = \{z \in \mathbb{C} : |z| < 1 \text{ or } |z - 3i| < 1\}$ is a domain.

__________ (ix) A boundary point of a set $S \subset \mathbb{C}$ is a point of S.

__________ (x) The complex plane with the real and imaginary axes deleted has no boundary points.

__________ (xi) If $f(z) = u(x, y) + iv(x, y)$ is analytic at z_0, then necessarily the function $g(z) = v(x, y) - iu(x, y)$ is analytic at z_0.

__________ (xii) $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{483} = -1$.

__________ (xiii) If $f(z) = e^x \cos(y) + ie^x \sin(y)$, then $f'(z) = f(z)$ for all $z \in \mathbb{C}$.

__________ (xiv) The domain of $f(z) = \dfrac{1}{z^2 + 1}$ is $\mathbb{C}$. .

__________ (xv) The entire complex plane is mapped into the real axis by the mapping $w = z + \bar{z}$.

__________ (xvi) All of $\mathbb{C}$ is mapped onto the unit circle by $w = \frac{z}{|z|}$.

__________ (xvii) The image of the disk $D(z_0, r)$ under a linear mapping $L(z) = az + b$, for $a, b \in \mathbb{C}$ is always a disk.

__________ (xviii) The linear mapping $w = L(z) = (1 - i\sqrt{3})z + 2$ acts by rotating through an angle of $\pi/3$ radians clockwise about the origin, magnifying by a factor of 2, then translating by 2.

__________ (xiv) Under the mapping $w = 1/z$, the extended complex plane, $\hat{C} = \mathbb{C} \cup \{\infty\}$, is mapped onto the disk $D(0, 1/3) = \{z \in \mathbb{C} : |z| < 1/3\}$.

__________ (xx) If g is an entire function, then so is $f(z) = (iz^2 + z)g(z)$.

Warm ups! Part 2: Short Response

(i) The real and imaginary parts of $f(z) = z^2 + i\bar{z}$ are

(ii) A parametrization of a line segment from $1 + i$ to $5i$ is

(iii) A (counter-clockwise) parametrization of the circle centered at $3 - 5i$ with radius 7 is

(iv) If $f(z) = \frac{z+\bar{z}}{z}$ compute $\lim_{z\to 0} f(z)$ along two paths: (i) z approaches 0 along the real axis; (ii) z approaches 0 along the imaginary axis; (iii) Does $\lim_{z\to 0} f(z)$ exist?

(v) Find $f'(z)$ given $f(z) = \dfrac{z+1}{z^2 + 5iz - 4}$

(vi) $\mathrm{Ln}(z)$ is discontinuous on

(vii) If $\mathrm{Ln}(z)$ is purely imaginary, what does that imply about z?

(viii) Find n such that $(1 + i)^n = 4096$

(ix) If $|e^z| = 1$ then what can you say about z?

(x) $i^{i^i} =$

Problem 1. $\mathbb{C}$ as points in the place.

Interpret complex numbers as points in the plane. Let $\lambda \in \mathbb{R}$ satisfying $2 < \lambda < 3$ and define $z = \lambda(1 + i)$. For each of the following parts draw a new $\mathbb{C}$-plane.

(a) Sketch z and $w = \bar{z}$ in the same complex plane $\mathbb{C}$.

(b) Sketch z and $w = \frac{1}{z}$ in the same complex plane $\mathbb{C}$.

(c) Sketch z and $w = z^2$ in the same complex plane $\mathbb{C}$.

(d) Sketch z and all the w satisfying $w^2 = z$ in the same complex plane $\mathbb{C}$.

Problem 2. Linear Mappings in $\mathbb{C}$.

Consider the complex linear function $L(z) = (1+i)z + 3 - 4i$. Let A be the square with vertices $V_1 = 1 + i$, $V_2 = -1 + i$, $V_3 = -1 - i$, and $V_4 = 1 - i$ including its boundary and interior points.

(a) Let $B = L(A)$, that is, the image of A under L. What is B? Describe the mapping $L : A \to B$ in words and graphically. [Hint: simply find the images of the vertices]

(b) A *fixed point* of a function f is a point z_0 that is not moved by the function, that is, $f(z_0) = z_0$. The function L has one fixed point. What is it?

(c) Describe the mapping L geometrically, that is, what does it do to points in $\mathbb{C}$?

(d) Draw $C = \{z \in \mathbb{C} : |z| \leq 1\}$, with a happy face (oriented in the usual way), and $D = L(C)$ the image of C under L. Describe the boundary of D.

Problem 3. Complex Exponential Mapping.

We'll study $f(z) = e^z$. In each part, draw a separate $\mathbb{C}$-plane for the pre-image (inputs z) and the image (outputs w) and include orientations.

(a) Graph the images of the horizontal lines $\{z \in \mathbb{C} : \text{Im}\,(z) = -\pi\}$, $\{z \in \mathbb{C} : \text{Im}\,(z) = -\frac{\pi}{2}\}$, $\{z \in \mathbb{C} : \text{Im}\,(z) = 0\}$, $\{z \in \mathbb{C} : \text{Im}\,(z) = \frac{\pi}{2}\}$, and $\{z \in \mathbb{C} : \text{Im}\,(z) = \pi\}$ [Bonus: use colors!].

(b) Graph the images of the vertical lines $\{z \in \mathbb{C} : \text{Re}\,(z) = \ln(1/2)\}$, $\{z \in \mathbb{C} : \text{Re}\,(z) = \ln(1)\}$, and $\{z \in \mathbb{C} : \text{Re}\,(z) = \ln(2)\}$ [Bonus: use colors!].

(c) Graph the pre-image and image of the principal branch of $g(z) = \text{Ln}(z)$.

Problem 4. Cube Roots Mapping.

We'll study $f(z) = z^{1/3}$. In each part, draw a separate $\mathbb{C}$-plane for the pre-image (inputs z) and the image (outputs w) and include orientations.

(a) Graph the images of the rays $\{z \in \mathbb{C} \setminus \{0\} : \text{Arg}(z) = 0\}$, $\{z \in \mathbb{C} \setminus \{0\} : \text{Arg}(z) = \pi/2\}$, and $\{z \in \mathbb{C} \setminus \{0\} : \text{Arg}(z) = \pi\}$ [Bonus: use colors!].

(b) Graph the images of the semi-circles $\{z \in \mathbb{C}\} : |z| = 1/8, \text{Im}\,(z) \geq 0\}$, $\{z \in \mathbb{C}\} : |z| = 1, \text{Im}\,(z) \geq 0\}$, and $\{z \in \mathbb{C}\} : |z| = 8, \text{Im}\,(z) \geq 0\}$ where here $\ln(x)$ is the usual real-valued natural logarithm [Bonus: use colors!].

(c) Graph the pre-image and image of the principal branch.

Problem 5. Analytic Functions.

(a) If $f(z) = (x^2 + y^2) + i(2xy)$. Find the set of z for which $f'(z)$ exists.

(b) Is f analytic on the set found in part (a)?

(c) Show $v = \text{Im}\,(f)$ is harmonic. Is $u = \text{Re}\,(f)$ the harmonic conjugate of v? If not, find it's harmonic conjugate.

(d) Is $S(z) = \sqrt{z}$ an analytic function on H, where $H = \{z \in \mathbb{C} : \text{Re}\,(z) > 0\}$? Justify your answer by computing $\dfrac{d}{dz}\left[\sqrt{z}\right]$ using the limit definition for $z \in H$.

Problem 6. Cauchy-Riemann Equations.

(a) If f is analytic on a domain $D \subset \mathbb{C}$ and $f'(z) = 0$ for all $z \in D$, show $f(z)$ is constant everywhere on D.

(b) If both f and $\bar{f}$ are analytic on a domain $D \subset \mathbb{C}$, show $f(z)$ is constant everywhere on D.

(c) If f is analytic on a domain $D \subset \mathbb{C}$ and $|f(z)| = c$ where c is a (real) constant, then $f(z)$ is constant everywhere on D. [Hint: break it up into two cases: (i) $c = 0$, (ii) $c \neq 0$. In case (ii), find a System of Equations in u_x and u_y]

Problem 7. Dot Products

Dot Products are one of the most important tools in mathematics, physics, and engineering. In this problem, we'll explore it's definition using the various perspectives complex variables allows. Let $z, w \in \mathbb{C}$ and interpret them as vectors.

Assume $|z| = 1$. Define the *dot product*, $z \bullet w$, to be the magnitude of the projection of w onto z, or $|proj_z(w)|$.

(a) Explain geometrically why $z \bullet w = |w| \cos(\theta)$, where θ is the angle between z and w.

(b) Explain geometrically using complex multiplication that $z \bullet w = \text{Re}\ (\bar{z}w)$.

(c) Explain geometrically using complex multiplication that $z \bullet w = \frac{1}{2}(\bar{z}w + z\bar{w})$.

(d) Find an expression of $z \bullet w$ in Cartesian coordinates using $z = x_1 + iy_1$ and $w = x_2 + iy_2$.

Now, define the dot product for any $z, w \in \mathbb{C}$ (that is, z is no longer assumed to be of unit length) by $z \bullet w = |z|\,|proj_z(w)| = |w|\,|proj_w(z)|$.

(e) Show that $z \bullet w = |z|\,|w|\ \cos(\theta)$.

(f) Do the expressions in part (b), (c), and (d) still hold?

Problem 8. Cross Products

Cross Products are one of the most important tools in mathematics, physics, and engineering. In this problem, we'll explore it's definition using the various perspectives complex variables allows. Let $z, w \in \mathbb{C}$ and interpret them as vectors.

Define the *cross product*, $z \times w$, to be the area of the parallelogram formed by z and w.

(a) Explain geometrically that $z \times w = |z|\,|w|\ \sin(\theta)$, where θ is the angle between z and w.

(b) Explain geometrically using complex multiplication that $z \times w = \text{Im}\ (\bar{z}w)$.

(c) Explain geometrically using complex multiplication that $z \times w = \frac{1}{2i}(\bar{z}w - z\bar{w})$.

(d) Find an expression of $z \times w$ in Cartesian coordinates using $z = x_1 + iy_1$ and $w = x_2 + iy_2$.

We now understand why $\bar{z}w$ is interesting since it encodes both the dot and cross products because $\bar{z}w = z \bullet w + i(z \times w)$.

Problem 9. Cauchy-Riemann Equations in Polar Coordinates

Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function on a domain $D \subset \mathbb{C}$. Denote by $\tilde{u}(r, \theta)$ and $\tilde{v}(r, \theta)$ the real and imaginary parts of f written in polar coordinates (r, θ), so we also have that $f(z) = \tilde{u}(r, \theta) + i\tilde{v}(r, \theta)$. Recall that the two coordinates are related by $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

(a) Find $\tilde{u}$ and $\tilde{v}$ for $f(z) = z^2$.

Now, recall that in multi-variable calculus, taking derivatives of a function of more than one variables, say $\phi : \mathbb{R}^2 \to \mathbb{R}$, in different coordinates works like this: view x as a function of r, θ, $x(r, \theta)$, and similarly, $y(r, \theta)$, then

$$\frac{\partial}{\partial r}\phi(x, y) = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial r}$$

and

$$\frac{\partial}{\partial\theta}\phi(x,y) = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial\theta}$$

(b) Using the above, compute $\tilde{u}_r$, $\tilde{u}_\theta$, $\tilde{u}_\theta$, and $\tilde{v}_\theta$.

(c) Using part (b), write the Cauchy-Riemann Equations for $\tilde{u}$ and $\tilde{v}$. (Note: these should include information only in r and θ).

Problem 10. Möbius Transformations

Möbius Transformations are complex functions of the form: $M : \mathbb{C} \to \mathbb{C}$ where

$$M(z) = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0.$$

In this problem, we'll investigate the specific Möbius Transformations $M(z) = \dfrac{1-z}{1+z}$.

(a) Show that $M(z) = 2\frac{1}{1+z} - 1$.
(b) Show that $M(z) = (f_3 \circ f_2 \circ f_1)(z)$, where $f_1(z) = 1 + z$, $f_2(z) = \frac{1}{z}$, and $f_3(z) = 2z - 1$.
(c) Is $(f_1 \circ f_2 \circ f_3)(z) = M(z)$ as well? What about all the other permutations?
(d) Let $A = \{z \in \mathbb{C} : |z| = 1\}$. Let $B = f_1(A)$. What is B? Describe the mapping $f_1 : A \to B$ in words and graphically.
(e) Let $C = f_2(B)$, that is, the image of B under f_2. What is C? Describe the mapping $f_2 : B \to C$ in words and graphically. If $0 \in B$, where does it get mapped to? [Hint: Recall that inversion $1/z$ maps vertical lines Re $(z) = c$ into circles $B = \{w \in \mathbb{C} : |w - \frac{1}{2c}| = \frac{1}{|2c|}\}$, for all $c \in \mathbb{R} \setminus \{0\}$.]
(f) Let $D = f_3(C)$, that is, the image of C under f_3. What is D? Describe the mapping $f_3 : C \to D$ in words and graphically.
(g) Use a series of pictures to describe the action of the mapping M on the unit circle A and include orientation in your drawings.
(h) The interior of A is the open disk $A^\circ = \{z \in \mathbb{C} : |z| < 1\}$. Where does M map the interior A°? [Hint: pick a point Z in A° and keep track of it via the mappings f_1, f_2, f_3.]

For Fun :-) Do not turn-in!

(a) Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a real-valued function of two variables, $\phi(x, y)$. We'll denote vectors in the plane in Cartesian coordinates by $\langle x, y \rangle$ to differentiate them from the point (x, y). Define the "gradient of " ϕ to be the vector $\nabla\phi = \langle \frac{\partial}{\partial x}\phi, \frac{\partial}{\partial x}\phi \rangle = \langle \phi_x, \phi_y \rangle$.

(i) Show that if f is an analytic function, then $|f'(z)| = |\nabla(\text{Re } f)| = |\nabla(\text{Im } f)|$.

(ii) Show that $\nabla(\text{Re } f)$ and $\nabla(\text{Im } f) = 0$ are perpendicular vectors, i.e. show that $\nabla(\text{Re } f) \bullet \nabla(\text{Im } f) = 0$ where $\bullet$ is the Dot Product defined in Problem 7.

(b) Fix any $a \in \mathbb{C}$. Show that $\dfrac{|z - a|}{|1 - \bar{a}z|} = 1$ if z is any point on the unit circle (i.e. $|z| = 1$) and $\bar{a}z \neq 1$. [Note: this important fact is related to the mappings in Problem 10.]

(c) Show that $f(z) = |z|^2$ is differentiable at $z = 0$ but not at any other point.