PEARSON'S

Calculus Review, Multivariable

PEARSON

Vectors and the Geometry of Space

THREE-DIMENSIONAL COORDINATE SYSTEMS

The distance between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

The standard equation for the sphere of radius a and center (x_0, y_0, z_0) is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

VECTORS

Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$, $\mathbf{v} = \langle v_1, v_2, v_3\rangle$, and $\mathbf{w} = \langle w_1, w_2, w_3\rangle$ be vectors and k a scalar.

The magnitude or length of a vector $\mathbf{v}$ is $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$. The vector with initial point $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ is $\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle$.

OPERATIONS ON VECTORS

Addition: $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3\rangle$

Scalar multiplication: $k\mathbf{u} = \langle ku_1, ku_2, ku_3\rangle$

Dot Product: $\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$

The angle θ between $\mathbf{u}$ and $\mathbf{v}$ is $\theta = \cos^{-1}\left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)$. Since $\cos\left(\dfrac{\pi}{2}\right) = 0$ we have that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal or perpendicular if and only if $\mathbf{u}\cdot\mathbf{v} = 0$. Also, $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2$.

The vector projection of $\mathbf{u}$ onto $\mathbf{v}$ is $\text{Proj}_{\mathbf{v}}\,\mathbf{u} = \left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$.

CROSS PRODUCT

$$\mathbf{u}\times\mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

CROSS PRODUCT PROPERTIES

1. $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$ = area of parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.
2. $\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})$, $\mathbf{u}\times\mathbf{u} = \mathbf{0}$
3. $\mathbf{i}\times\mathbf{j} = \mathbf{k}$, $\mathbf{j}\times\mathbf{k} = \mathbf{i}$, $\mathbf{k}\times\mathbf{i} = \mathbf{j}$
4. $\mathbf{u}\times\mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, thus $\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot(\mathbf{u}\times\mathbf{v}) = 0$.
5. $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ = volume of parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

LINES AND PLANES IN SPACE

A vector equation for the line L through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v}$ is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \quad -\infty < t < \infty$$

where $\mathbf{r}$ is the position vector of a point $P(x, y, z)$ on L and $\mathbf{r}_0$ is the position vector of $P_0(x_0, y_0, z_0)$.

The plane through $P_0(x_0, y_0, z_0)$ normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has

vector equation:

$$\mathbf{n}\cdot\overrightarrow{P_0P} = 0$$

component equation:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

component equation simplified:

$$Ax + By + Cz = D, \text{ where } D = Ax_0 + By_0 + Cz_0$$

QUADRIC SURFACES

1. Ellipsoid: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$
2. Elliptical Paraboloid: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z}{c}$
3. Elliptical Cone: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2}$

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4. Hyperboloid of one sheet: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$
5. Hyperboloid of two sheets: $\dfrac{z^2}{c^2} - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$
6. Hyperbolic Paraboloid: $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = \dfrac{z}{c}$

Vector-Valued Functions and Motion in Space

Let $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\mathbf{k}$ be a vector function. Then $\mathbf{v}(t) = \dfrac{d\mathbf{r}}{dt}$ is the velocity vector and $|\mathbf{v}(t)|$ is the speed. The acceleration vector is $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{d^2\mathbf{r}}{dt^2}$. The unit tangent vector is $\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$ and the length of $\mathbf{r}(t)$ from $t = a$ to $t = b$ is

$$L = \int_a^b |\mathbf{v}|\,dt.$$

PRINCIPAL UNIT NORMAL

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$$

BINORMAL

$$\mathbf{B} = \mathbf{T}\times\mathbf{N}$$

FORMULAS

Curvature: $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \dfrac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \dfrac{1}{|\mathbf{v}|}\left|\dfrac{d\mathbf{T}}{dt}\right|$

Radius of Curvature: $\rho = \dfrac{1}{\kappa}$

Torsion: $\tau = -\dfrac{d\mathbf{B}}{ds}\cdot\mathbf{N} = \dfrac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2}$

Tangential and normal scalar components of acceleration:

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$$

where

$$a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}|\mathbf{v}|$$

and

$$a_N = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}.$$

Partial Derivatives

To compute $\dfrac{\partial f}{\partial x}$, differentiate $f(x, y)$ thinking of y as a constant.

To compute $\dfrac{\partial f}{\partial y}$, differentiate $f(x, y)$ thinking of x as a constant.

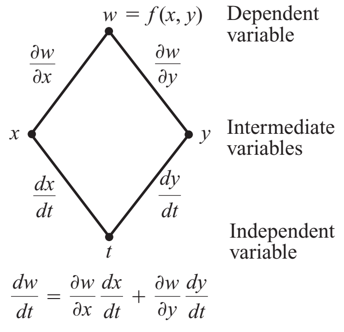
Thus, if $f(x, y) = y\cos(xy)$, $\dfrac{\partial f}{\partial y} = -y^2\sin xy$, and $\dfrac{\partial f}{\partial y} = \cos xy - xy\sin xy$.

NOTATION

$$\frac{\partial f}{\partial x} = f_x, \quad \frac{\partial f}{\partial y} = f_y$$

CHAIN RULE

To find dw/dt, start at w and read down each route to t, multiplying derivatives along the way. Then add the products.



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

GRADIENT VECTOR

The gradient vector (gradient) of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

obtained by evaluating the partial derivatives of f at P_0.

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DIRECTIONAL DERIVATIVE

The directional derivative of $f(x, y)$ at P_0 in direction of unit vector $\mathbf{u}$ is $\left(\dfrac{df}{ds}\right)_{\mathbf{u}, P_0} = (\nabla f)_{P_0}\cdot\mathbf{u}$ (the dot product of $\mathbf{u}$ and ∇f).

It is also denoted by $(D_{\mathbf{u}}f)_{P_0}$.

TANGENT PLANE AND NORMAL LINE TO A SURFACE

The tangent plane at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ is the plane through P_0 normal to $\nabla f|_{P_0}$.

The normal line of the surface at P_0 is the line through P_0 parallel to $\nabla f|_{P_0}$.

Tangent Plane to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$

Normal Line to $f(x, y, z) = c$

$$x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t$$

Plane Tangent to a Surface $z = f(x, y)$

The plane tangent to the surface $z = f(x, y)$ at the point $P_0(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

LINEARIZATION

The linearization of a function $f(x, y)$ at a point (x_0, y_0) where f is differentiable is

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

SECOND DERIVATIVE TEST FOR LOCAL EXTREMA

Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

- i. f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).
- ii. f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).
- iii. f has a **saddle point** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b).
- iv. **The test is inconclusive** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b). In this case, we must find some other way to determine the behavior of f at (a, b).

LAGRANGE MULTIPLIERS

One Constraint: Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable. To find the local maximum and minimum values of f subject to the constraint $g(x, y, z) = 0$, find the values of x, y, z, and λ that simultaneously satisfy the equations

$$\nabla f = \lambda\nabla g \quad \text{and} \quad g(x, y, z) = 0.$$

Two Constraints: For constraints $g_1(x, y, z) = 0$ and $g_2(x, y, z) = 0$, g_1 and g_2 differentiable, find the values of x, y, z, λ, and μ that simultaneously satisfy the equations

$$\nabla f = \lambda\nabla g_1 + \mu\nabla g_2, \quad g_1(x, y, z) = 0, \quad g_2(x, y, z) = 0.$$

Multiple Integrals

DOUBLE INTEGRALS AS VOLUMES

When $f(x, y)$ is a positive function over a region R in the xy-plane, we may interpret the double integral of f over R as the volume of the 3-dimensional solid region over the xy-plane bounded below by R and above by the surface $z = f(x, y)$.

This volume can be evaluated by computing an iterated integral.

Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then
$$\iint_R f(x, y)\,dA = \int_a^b\int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx.$$
2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then
$$\iint_R f(x, y)\,dA = \int_c^d\int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy.$$

AREA VIA DOUBLE INTEGRAL

$$A = \iint_R dA = \iint_R dx\,dy = \iint_R dy\,dx$$

AREA IN POLAR COORDINATES

$$A = \iint_R r\,dr\,d\theta$$

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TRIPLE INTEGRALS

$$\iiint_D F(x, y, z)\, dV = \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\, dz\, dy\, dx$$

if D can be described as the solid in space with $f_1(x, y) \le z \le f_2(x, y)$ over the region in the xy-plane with $a \le x \le b$, $g_1(x) \le y \le g_2(x)$.

CYLINDRICAL COORDINATES

Equations Relating Rectangular x, y, z and Cylindrical (r, θ, z) Coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z,$$

$$r^2 = x^2 + y^2, \quad \tan\theta = y/x$$

TRIPLE INTEGRALS IN CYLINDRICAL COORDINATES

$$\iiint_D f(r, \theta, z)\, dV = \iiint_D f(r, \theta, z)\, dz\, r\, dr\, d\theta$$

SPHERICAL COORDINATES (ρ, ϕ, θ)

Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

$$r = \rho\sin\phi \qquad x = r\cos\theta = \rho\sin\phi\cos\theta,$$

$$z = \rho\cos\phi \qquad y = r\sin\theta = \rho\sin\phi\sin\theta,$$

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$$

TRIPLE INTEGRALS IN SPHERICAL COORDINATES

$$\iiint_D f(\rho, \phi, \theta)\, dV = \iiint_D f(\rho, \phi, \theta)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

CHANGE OF VARIABLES FORMULA FOR DOUBLE INTEGRALS

$$\iint_R f(x, y)\, dy\, dx = \iint_R f(g(u, v), h(u, v))\, |J(u, v)|\, du\, dv$$

where $x = g(u, v)$, $y = h(u, v)$ take region G onto region R and

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

is the Jacobian determinant $\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \frac{\partial(x, y)}{\partial(u, v)}$.

Integration in Vector Fields

LINE INTEGRALS

To integrate a continuous function $f(x, y, z)$ over a curve C:

1. Find a smooth parametrization of C,

   $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}, \quad a \le t \le b$

2. Evaluate the integral as

$$\int_C f(x, y, z)\, ds = \int_a^b f(g(t), h(t), k(t))\, |\mathbf{v}(t)|\, dt$$

   where $\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$ and $ds = |\mathbf{v}(t)|\, dt$.

WORK

The work done by a force $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ over a smooth curve $\mathbf{r}(t)$ from $t = a$ to $t = b$ is

$$W = \int_{t=a}^{t=b} \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_a^b \mathbf{F} \cdot d\mathbf{r} = \int_a^b M dx + N dy + P dz.$$

FLOW

$$\text{Flow} = \int_a^b \mathbf{F} \cdot \mathbf{T}\, ds$$

FLUX

Flux of $\mathbf{F}$ across $C = \int_C \mathbf{F} \cdot \mathbf{n}\, ds = \oint_C M dy - N dx$ where

$\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ and $\mathbf{n}$ is outward pointing normal along C.

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CONSERVATIVE VECTOR FIELD

$\mathbf{F}$ is conservative if $\mathbf{F} = \nabla f$ for some function $f(x, y, z)$. If $\mathbf{F}$ is conservative, then

$$\int_A^B \mathbf{F} \cdot d\mathbf{r}$$

is independent of path and

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

Also,

$$\int \mathbf{F} \cdot d\mathbf{r} = 0$$

around every closed loop in this case.

COMPONENT TEST FOR CONSERVATIVE FIELDS

Let $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ be a field whose component functions have continuous first partial derivatives on an open, simply connected region. Then $\mathbf{F}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

GREEN'S THEOREM

If C is a simple closed curve and R is the region enclosed by C then

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \oint_C M dy - N dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy$$

Outward flux — Divergence integral

and

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

Counterclockwise circulation — Curl integral

GREEN'S THEOREM AREA FORMULA

$$\text{Area of } R = \frac{1}{2}\oint_C x\, dy - y\, dx$$

SURFACE INTEGRALS

Let $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$

$a \le u \le b$, $c \le v \le d$ be a parametrization of a surface S.

Let $\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} = f_u\mathbf{i} + g_u\mathbf{j} + h_u\mathbf{k}$ and $\mathbf{r}_v = f_v\mathbf{i} + g_v\mathbf{j} + h_v\mathbf{k}$.

The unit normal to the surface is

$$\frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} = \mathbf{n}.$$

AREA

The area of the surface S is $\int_c^d \int_a^b |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv$.

SURFACE INTEGRAL

If $G(x, y, z)$ is defined over S then the integral of G over S is

$$\iint_S G(x, y, z)\, d\sigma = \int_c^d \int_a^b G(f(u, v), g(u, v), h(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv$$

where

$$d\sigma = |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv.$$

CURL OF A VECTOR FIELD

If $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ then

$$\text{curl } \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} = \nabla \times \mathbf{F}$$

DIVERGENCE OF A VECTOR FIELD

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

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STOKES' THEOREM

The circulation of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{j}$ around the boundary C of an oriented surface S in the direction counterclockwise with respect to the surface's unit normal vector $\mathbf{n}$ equals the integral of $\nabla \times \mathbf{F} \cdot \mathbf{n}$ over S.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

Counterclockwise circulation — Curl integral

DIVERGENCE THEOREM

The flux of a vector field $\mathbf{F}$ across a closed oriented surface S in the direction of the surface's outward unit normal field $\mathbf{n}$ equals the integral of $\nabla \cdot \mathbf{F}$ over the region D enclosed by the surface:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV.$$

Outward flux — Divergence integral

GREEN'S THEOREM AND ITS GENERALIZATION TO THREE DIMENSIONS

Normal form of Green's Theorem:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \nabla \cdot \mathbf{F}\, dA$$

Divergence Theorem:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV$$

Tangential form of Green's Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k}\, dA$$

Stokes' Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

IMPROPER INTEGRALS

Integrals with infinite limits of integration are improper integrals of Type I.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\, dx = \lim_{b\to\infty} \int_a^b f(x)\, dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\, dx = \lim_{a\to-\infty} \int_a^b f(x)\, dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\, dx = \int_{-\infty}^c f(x)\, dx + \int_c^\infty f(x)\, dx,$$

   where c is any real number.

Integrals of functions that become infinite at a point within the interval of integration are improper integrals of Type II.

4. If $f(x)$ is continuous on $(a, b]$ and is discontinuous at a then

$$\int_a^b f(x)\, dx = \lim_{c\to a^+} \int_c^b f(x)\, dx.$$

5. If $f(x)$ is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\, dx = \lim_{c\to b^-} \int_a^c f(x)\, dx.$$

6. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

In each case, if the limit is finite we say the improper integral converges and that the limit is the value of the improper integral. If the limit does not exist, the integral diverges.