MATH 5B - Single Variable Calculus II — Spring 2019

Hand-out

Eight Definitions of "e"

Dr. Jorge Basilio

Pasadena City College

gbasilio@pasadena.edu

Introduction

Exponential functions of the form, $f(x) = b^x$, where b is a fixed real number that is positive and never one, are used for many applications throughout the "real world." Population growth, Radioactive decay are the main examples but Carbon-dating and Earthquake detections also use these functions.

Among the many exponential functions, like $f(x) = 2^x$ or $f(x) = (\frac{1}{3.1})^x$, there's one base that stands out as the most important. This base is an irrational number approximately equal to 2.718 and is referred to by the letter "e." Unfortunately, e is an irrational number (an infinitely non-repeating decimal) so giving a precise answer to it's digits is impossible. In fact, below I will give you six different definitions of the number "e" and all of the require Calculus. Many of them use the idea of a "limit" which we will study soon in the next chapter.

Interestingly, I believe the number "e" was first discovered by a mathematician interested in finance and debt. Who says the business world never contributed anything useful to humanity.

Compounding Interest

Suppose that you deposit P dollars into an account which computes interest at a rate of r% where interest is compounded n times each year for t years. The formula for the amount A can be shown to be:

$$A = P \cdot \left(1 + \frac{r}{n}\right)^{n \cdot t}$$

When $n = 1$, that is, interest is compounded once a year, this formula is called the simple interest formula and takes the form:

$$A = P \cdot (1 + r)^t$$

What happens when interest is compounded n times with n growing larger and larger? Each time we compound interest the amount grows. Intuitively, if we compound interest to larger and larger number of times each year, then we expect our principal to keep growing and growing. And it does, however, the amount that it grows by slows down and at some point becomes useless for finance purposes because we only care about the first two decimal places.

Take a look at the table provided in definition (1) of "e" from below. This formula comes from the formula of compounding interest provided that $P = \$1$ is invested at an interest rate of 100% (which is $r = 1$ when written as a decimal) after one year ($t = 1$).

If we compound interest once a year ($n = 1$), at a rate of 100%, then our investment of \$1 has grown to \$2. If we compound interest twice a year ($n = 2$), then our investment of \$1 has grown to \$2.25. So compounding interest twice gives us 25 cents more! If we compound interest ten times a year ($n = 10$), then our investment of \$1 has grown to \$2.59 or so. Again, the more times we compound interest the more money we get. Great!

However, the amount of money we gain starts to diminish. Even after compounding interest one thousand times in a single year ($n = 1000$), our dollar grew only to \$2.71. Not a dramatic improvement from \$2.59. Well, what if we compound interest 100,000 times? We get: 2.718268237. . .. So, we only improved by \$0.002 dollars, not really worth it from a computational point of view.

(1) Continuous Compounding Interest

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

n	$(1+\frac{1}{n})^n$
1	2
2	2.25
10	2.59374
20	2.65329
100	2.70481
1000	2.71692

(2) Continuous Compounding Interest (Alternative)

$$e = \lim_{h\to 0^+}(1+h)^{1/h}$$

h	$(1+h)^{1/h}$
1	2
0.5	2.25
0.1	2.59374
0.05	2.65329
0.01	2.70481
0.001	2.71692

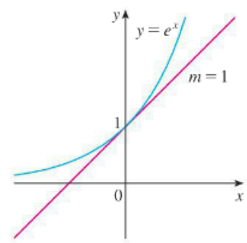
(3) INFINITE SUM

$$e = \sum_{n=0}^{\infty}\frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{3\cdot 2\cdot 1} + \frac{1}{4\cdot 3\cdot 2\cdot 1} + \cdots$$

n	Sum of first n terms
1	1
2	2
3	2.5
4	2.66667
5	2.70833
10	2.71828

(4) GEOMETRIC TANGENT LINE

$e =$ the number so that the exponential function with this base has a tangent line with slope 1 at the point $(0, 1)$



(5) CONTINUED FRACTION

$e =$ the unique number so that

$$e = 2 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\cfrac{4}{\ddots}}}}}$$

(6) DERIVATIVE

$e =$ the unique number so that the limit is true

$$1 = \lim_{h\to 0}\frac{e^h - 1}{h}$$

(7) DIFFERENTIAL EQUATIONS

$e =$ the base of the exponential function $y = b^x$ satisfying the differential equation with $y(0) = 1$

$$\frac{dy}{dx} = y$$

(8) INTEGRAL

$e =$ the number so that the area under $y(x) = \frac{1}{x}$ from $x = 1$ to $x = e$ equals 1:

$$1 = \int_1^e \frac{1}{t}\,dt$$

How are they related?

- (1) is equivalent to (2): to see this make the substitution $n = 1/h$
- (4) is equivalent to (6): easy exercise.
- Definition (3) will be proven later in the semester
- Definition (5) involves more advanced mathematics but I think it looks cool so I've included it :-)
- Definiton (7) is equivalent to (4) and (6)
- Definition (8) is a common definition based on the natural logarithm. If we define $\ln(x) = \int_1^x \frac{1}{t}dt$ then (8) says: $\ln(e) = 1$.