What is Multivariable Calculus?

A Brief History of Mathematics and Calculus

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Summer 2019

Outline

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- **Ancient Problem 3**: What is a (real) Number? (Egypt, Mesopotamia, Greek)

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- Ancient Problem 1: Area under a curve (Egypt, Mesopotamia, Greek)
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- Ancient Problem 3: What is a (real) Number? (Egypt, Mesopotamia, Greek)
- Ancient Problem 4: Tangent Line Problem (Greek, Europe Middle Ages)

How were these problems solved?

- Calculus: "a PROCESS developed to solve hard problems in the following steps:"
  - Step 1: find an approximate solution to the hard problem
  - Step $1\frac{1}{2}$: find a better approximate solution, & then a better one, & a better, ETC
  - Step 2: the exact (ideal) answer = LIMIT of approximate solutions"
- Open Stewart's "A Preview of Calculus"
  - AP1: Area of a circle via Archimedes' "Method of Exhaustion"
  - AP4: Tangent Line Problem
  - AP2: Instantaneous Velocity
  - Limit of a sequence (related to Zeno?s Paradox/AP2 and also irrational numbers/AP3)
  - Sum of a series (related to irrational numbers/AP3)

Many different questions, the same approach towards a solution
BUT MUST DEAL WITH **LIMITS/INFINITY**

A Fear of Infinifty

Infinity has captivated mathematicians, philosophers, and poets like no other concept. It is as alluring as it is wicked.

The rules for how to correctly work with the concept are not obvious (unlike the rules for Euclidean geometry).

It took a long time to understand what is allowed and what is not allowed in infinite processes. The result of these studies is calculus, which you will now learn.

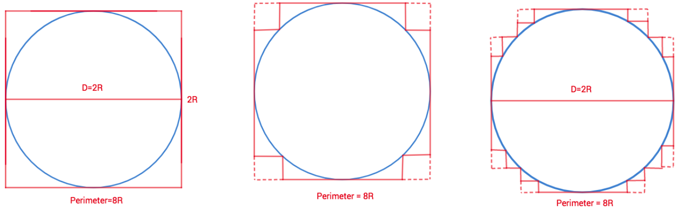
Here's two examples showing what can go wrong:

- **Zeno's Paradoxes** stumped the best minds for thousands of years
- $\pi = 4$

Theorem 1:

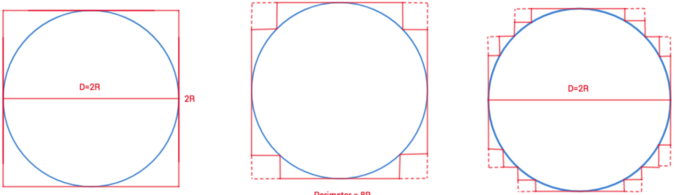
$\pi = 4$

Start with a circle of radius R. We first estimate the perimeter by $8R$ using a circumscribed square. Cut out four corners and use the new shorter sides to approximate the perimeter. Notice that the perimeter is still $8R$. Approximate the circle with more and more with sides parallel to axes. The perimeter is still $8R$. Limit of these jagged curves approximates the circle so the limit of these perimeter is the circumference of the circle. Thus, perimeter of full circle is $8R$. Because $C = 2\pi R$ we have $8R = 2\pi R$. We conclude: $4 = \pi$! □



Of course this is wrong!

There's a flaw in this argument that's *very* difficult to catch.
Can you find it?



Calculus Highlights

Briefly: Two branches of Calculus

1. Differential Calculus (derivatives: local to global)
   - $f'(x) = \dfrac{df}{dx} = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$
   - Applications: slope of a curve (tangent line), approximating a function with a tangent line, velocities & accelerations, rates of change, finding the minimum/maximum of a function (optimization)
2. Integral Calculus (integrals: infinite sums)
   - $\displaystyle\int_a^b f(x)dx = \lim_{n \to \infty} \left(\sum_{i=1}^{n} f(c_i)\Delta x \right)$
   - Applications: area under a curve, solving differential equations, volumes of solids of revolutions, arclength of a function

Both made possible thanks to limits

Calculus Highlights

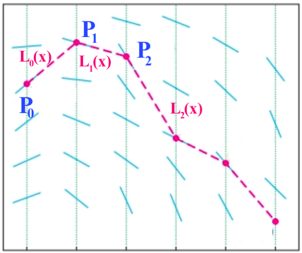
THEMES

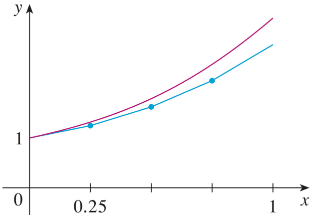
- Derivative Rules: short-cuts to limits
- Techniques for integration: short-cuts to limits
- Linearization: approximating a curve with the tangent line (="line of best fit")
- Min/Max Problems: Optimization
- Using integrals for geometry: lengths, areas, volumes
- Calculus: transform difficult **non-linear problems** into easier **linear** problems!!!!
- As an example: using Euler's method for solving differential equations.

Euler's Method: Solve $\frac{dy}{dx} = F(x, y)$.

Euler's Method

- Approximate solutions to DE with line segments using the tangent line approximations where the slopes are given by $m = y' = F(x, y)$
- Starting at the initial conditions, $P_0 = (x_0, y(x_0))$, construct the tangent line $L_0(x)$ with point P_0 and slope $m = F(x_0, y_0)$
- We move along this line by taking a step of size h to arrive at a new point $P_1 = (x_1, y_1)$.
- Once we're at P_1 we use the slope $F(x_1, y_1)$ to construct a new tangent line $L_1(x)$
- Move along $L_1(x)$ by taking another step of size h to arrive at P_2
- Do this as many times as needed





Definition 1: Definite Integrals

- Start with a continuous function f on a closed interval $[a, b]$
- Choose $n \geq 1$. Cut $[a, b]$ into n equal pieces: $\boxed{\Delta x = \dfrac{b-a}{n}}$.
- Create subintervals $I_i = [x_{i-1}, x_i]$ using $\boxed{x_i = a + i\Delta x}$.
- Either pick a random $c_i \in [x_{i-1}, x_i]$ (or more systematically, like left-endpoints, midpoints, etc).
- The area of each sub-rectangle is $f(c_i)\Delta x$. For Δx small, these approximate the area under the graph of f.
- Sum the approximations: $\sum_{i=1}^{n} f(c_i)\Delta x$.
- The exact area is the limit: $\displaystyle\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$

Theorem 2: Fundamental Theorem of Calculus

Part 1 If f is a continuous function on the closed interval $[a, b]$, then

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Part 2 If f is defined on an interval I and F is an anti-derivative F of f also defined on I, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Remarks

- **Part 1** This says: the derivative of an integral of a function is the same as the original function.

  Or: "they undo each other!" Or: "the derivative and the integral are inverse operations" (like + and -)

- **Part 2** This is extremely useful for computations! Since we know a bunch of derivative rules, we can find anti-derivatives for many functions easily.

  Another reason why Part 2 is so useful–we complete skip over the true definition of an integral! We avoid hard limits and Riemann sums altogether! Horray!

Integration Toolbox

When confronted with an integral, $\int f(x)\,dx$, the main tools in your integration toolbox are:

1. know a lot of derivative rules/anti-derivative rules!
2. u-substitution (corresponds to the chain rule)
3. integration by parts (corresponds to the product rule)
4. trigonometric substitution

Additional techniques:

1. Strategies for $\sin^n(x) \cdot \cos^m(x)$
2. Partial Fractions
3. Miscellaneous algebra manipulations Note: sometimes this is "step 0"

What is Calculus III?

Calculus III:

- redo all of precalculus (analytic geometry)
  in 2D, 3D, and Higher-dimensions
- redo all of differential calculus
  in 2D, 3D, and Higher-dimensions
- redo all of integral calculus
  in 2D, 3D, and Higher-dimensions
- And study applications: linearization, min/max/optimization
- NEW!! Vector Analysis: new types of derivatives and integrals with applications to physics and engineering

What is Calculus III?

Overview & Goals

1. Analytic Geometry: study 3-dimensional space using coordinates $(x, y, z) \in \mathbb{R}^3$

   - lines, planes, spheres, surfaces

   - these are all conveniently expressed in the language of vectors

2. Generalize functions to several variables & redo "calculus 1 & 2"

   - limits, derivatives (partial derivatives, gradients) , integrals (double integrals, triple integrals)

3. Applications: linearization (tangent plane), min/max, surface areas, volumes in space

4. Vector Calculus

   - Curves and Surfaces in vector notation. Vector-valued functions, Vector fields

   - Line integrals, surface integrals with direction ("orientation")

What is Calculus III?

Functions in several variables

- Recall ordinary function: $f(x)$. This means: given an input $x \in \mathbb{R}$, $f(x)$ is a new real number called the output. Ex: $f(x) = x^2$, $f(-2) = 4$.
- How to generalize to several variables? Lots of ways to do this!
- Way #1: "scalar functions:" many inputs, one output. Graph is a surface in space.

  Ex: $f(x, y) = x^2 + y^2$. $f(-2, 1) = (-2)^2 + (1)^2 = 5$. Input: $(-2, 1)$, output:5
- Way #2: "space curves:" one input, many outputs. Graph is a space curve.

  Ex: $\vec{f}(t) = \langle \cos(t), \sin(t), t \rangle$. Input: t, outputs: $x = \cos(t)$, $y = \sin(t)$, $z = t$
- Way # 3: "vector-valued functions" (or "vector fields") many inputs, many outputs.

  Ex: $\vec{f}(x, y) = \langle -y, x \rangle$. Inputs: (x, y). Outputs: $x = -y$, $y = x$

Functions in several variables

- Summary: $\boxed{\vec{f} : \mathbb{R}^n \to \mathbb{R}^m}$ with $n \geq 1$ and $m \geq 1$.

$$\boxed{\vec{f}(x_1, x_2, \ldots, x_n) = \langle\, f_1(x_1, x_2, \ldots, x_n), f_2(x_1, x_2, \ldots, x_n), \ldots, f_m(x_1, x_2, \ldots, x_n)\,\rangle}$$

- Way #1: $m = 1$. "scalar functions" many inputs, one output. Graph is a surface in space.
- Way #2: "space curves:" $n = 1$, $m > 1$. one input, many outputs. Graph is a curve in space.
- Way # 3: "vector-valued functions" $n > 1$, $m > 1$. many inputs, many outputs.

Don't worry! We'll focus on 2D and 3D so only 3 variables (most of the time :P)

THANK YOU FOR YOUR ATTENTION