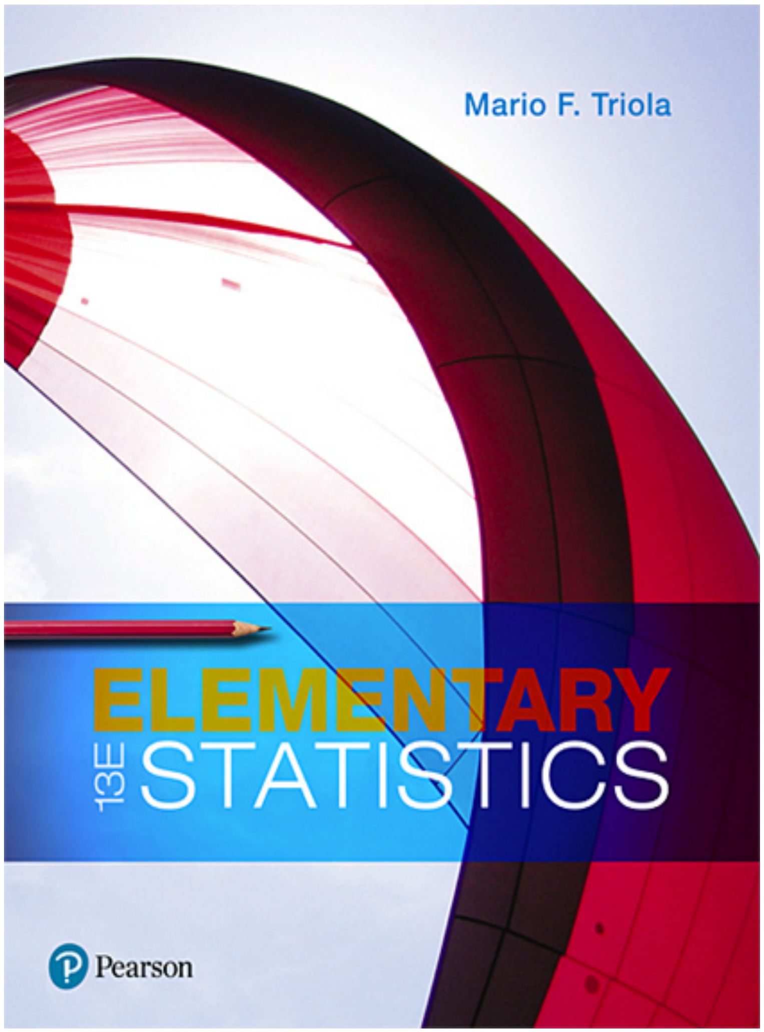
Elementary Statistics

Thirteenth Edition



Chapter 4

Probability

Probability

4-1 Basic Concepts of Probability

4-2 Addition Rule and Multiplication Rule

4-3 Complements and Conditional Probability, and Bayes' Theorem

4-4 Counting

4-5 Probabilities Through Simulations (available at TriloaStats.com)

Key Concept

This section requires much greater use of formulas as we consider five different methods for counting the number of possible outcomes in a variety of situations. Not all counting problems can be solved with these five methods, but they do provide a strong foundation for the most common real applications.

Multiplication Counting Rule

1. Multiplication Counting Rule
   – For a sequence of events in which the first event can occur n_1 ways, the second event can occur n_2 ways, the third event can occur n_3 ways, and so on, the total number of outcomes is $n_1 \cdot n_2 \cdot n_3 \ldots$.

Example: Multiplication Counting Rule: Hacker Guessing a Passcode (1 of 2)

When making random guesses for an unknown four-digit passcode, each digit can be 0, 1. . . , 9. What is the total number of different possible passcodes? Given that all of the guesses have the same chance of being correct, what is the probability of guessing the correct passcode on the first attempt?

Example: Multiplication Counting Rule; Hacker Guessing a Passcode (2 of 2)

Solution

There are 10 different possibilities for each digit, so the total number of different possible passcodes is $n_1 \cdot n_2 \cdot n_3 \cdot n_4 = 10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$.

Because all of the passcodes are equally likely, the probability of getting the correct passcode on the first attempt is $\frac{1}{10{,}000}$ or 0.0001.

Factorial Rule (1 of 2)

2. Factorial Rule
   - The number of different **arrangements** (order matters) of n different items when all n of them are selected is $\boldsymbol{n}$**!**.

Factorial Rule (2 of 2)

Notation

The **factorial symbol (!)** denotes the product of decreasing positive whole numbers. By special definition, $0! = 1$. The factorial rule is based on the principle that the first item may be selected n different ways, the second item may be selected $n - 1$ ways, and so on. This rule is really the multiplication counting rule modified for the elimination of one item on each selection.

Example: Factorial Rule: Travel Itinerary (1 of 3)

A statistics researcher must personally visit the presidents of the Gallup, Nielsen, Harris, Pew, and Zogby polling companies.

a. How many different travel itineraries are possible?

b. If the itinerary is randomly selected, what is the probability that the presidents are visited in order from youngest to oldest?

Example: Factorial Rule; Travel Itinerary (2 of 3)

Solution

a. For those 5 different presidents, the number of different travel itineraries is $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

Note that this solution could have been done by applying the multiplication counting rule. The first person can be any one of the 5 presidents, the second person can be any one of the 4 remaining presidents, and so on. The result is again $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

Example: Factorial Rule; Travel Itinerary (3 of 3)

Solution

b. There is only one itinerary with the presidents visited in order of age, so the probability is $\frac{1}{120}$

Permutations and Combinations: Does Order Count? (1 of 2)

- Permutations
  - **Permutations** of items are arrangements in which different sequences of the same items are counted **separately.** (The letter arrangements of abc, acb, bac, bca, cab, and cba are all counted **separately** as six different permutations.)

Permutations and Combinations: Does Order Count? (2 of 2)

- Combinations
  - **Combinations** of items are arrangements in which different sequences of the same items are counted as being the **same**. (The letter arrangements of abc, acb, bac, bca, cab, and cba are all considered to be the **same** combination.)

Mnemonics of Permutations and Combinations

- Remember “**P**ermutations **P**osition,” where the alliteration reminds us that with permutations, the positions of the items makes a difference.
- Remember “**C**ombinations **C**ommittee,” which reminds us that with members of a committee, rearrangements of the same members result in the same committee, so order does not count.

Permutations Rule (1 of 2)

3. Permutations Rule
   – When n different items are available and r of them are selected without replacement, the number of different permutations (order counts) is given by

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

Example: Permutations Rule (with Different Items): Trifecta Bet (1 of 4)

In a horse race, a **trifecta** bet is won by correctly selecting the horses that finish first and second and third, and you must select them in the correct order. The 140th running of the Kentucky Derby had a field of 19 horses.

Example: Permutations Rule (with Different Items): Trifecta Bet (2 of 4)

a. How many different trifecta bets are possible?

b. If a bettor randomly selects three of those horses for a trifecta bet, what is the probability of winning by selecting California Chrome to win, Commanding Curve to finish second, and Danza to finish third, as they did? Do all of the different possible trifecta bets have the same chance of winning? (Ignore “dead heats” in which horses tie for a win.)

Example: Permutations Rule (with Different Items): Trifecta Bet (3 of 4)

Solution

a. There are n = 19 horses available, and we must select r = 3 of them without replacement. The number of different sequences of arrangements is found a shown:

$$ {}_{n}P_{r} = \frac{n!}{(n-r)!} = \frac{19!}{(19-3)!} = 5814 $$

Example: Permutations Rule (with Different Items): Trifecta Bet (4 of 4)

Solution

There are 5814 different possible arrangements of 3 horses selected from the 19 that are available. If one of those arrangements is randomly selected, there is a probability of $\frac{1}{5814}$ that the winning arrangement is selected.

There are 5814 different possible trifecta bets, but not all of them have the same chance of winning, because some horses tend to be faster than others. (A winning $2 trifecta bet in this race won $3424.60.)

Permutations Rule (2 of 2)

4. Permutations Rule (When Some Items Are Identical to Others)

   The number of different permutations (order counts) when *n* items are available and all *n* of them are selected **without replacement,** but some of the items are identical to others, is found as follows:

$$\frac{n!}{n_1!n_2!\ldots n_k!}$$

where n_1 are alike, n_2 are alike, . . . , and n_k are alike.

Example: Permutations Rule (with Some Identical Items): Designing Surveys (1 of 3)

When designing surveys, pollsters sometimes repeat a question to see if a subject is thoughtlessly providing answers just to finish quickly. For one particular survey with 10 questions, 2 of the questions are identical to each other, and 3 other questions are also identical to each other. For this survey, how many different arrangements are possible? Is it practical to survey enough subjects so that every different possible arrangement is used?

Example: Permutations Rule (with Some Identical Items): Designing Surveys (2 of 3)

Solution

We have 10 questions with 2 that are identical to each other and 3 others that are also identical to each other, and we want the number of permutations. Using the rule for permutations with some items identical to others, we get

$$\frac{n!}{n_1!n_2!\ldots nk!} = \frac{10!}{2!3!} = \frac{3,628,800}{(2 \cdot 6)} = 302,400$$

Example: Permutations Rule (with Some Identical Items): Designing Surveys (3 of 3)

Interpretation

There are 302,400 different possible arrangements of the 10 questions. It is not practical to accommodate every possible permutation. For typical surveys, the number of respondents is somewhere around 1000.

Combinations Rule

5. Combinations Rule:

   When n different items are available, but only r of them are selected **without replacement,** the number of different combinations (order does not matter) is found as follows:

$$_{n}C_{r} = \frac{n!}{(n-r)!r!}$$

Example: Combinations Rule: Lottery (1 of 3)

In California's Fantasy 5 lottery game, winning the jackpot requires that you select 5 different numbers from 1 to 39, and the same 5 numbers must be drawn in the lottery. The winning numbers can be drawn in any order, so order does not make a difference.

a. How many different lottery tickets are possible?

b. Find the probability of winning the jackpot when one ticket is purchased.

Example: Combinations Rule: Lottery (2 of 3)

Solution

a. There are $n = 39$ different numbers available, and we must select $r = 5$ of them without replacement (because the selected numbers must be different). Because order does not count, we need to find the number of different possible **combinations**. We get

$$_{n}C_{r} = \frac{n!}{(n-r)!r!} = \frac{39!}{(39-5)!5!} = \frac{39!}{34! \cdot 5!} = 575{,}757$$

Example: Combinations Rule: Lottery (3 of 3)

Solution

b. If you select one 5-number combination, your probability of winning is $\frac{1}{575,757}$. Typical lotteries rely on the fact that people rarely know the value of this probability and have no realistic sense for how small that probability is. This is why the lottery is often called a “tax on people who are bad at math.”