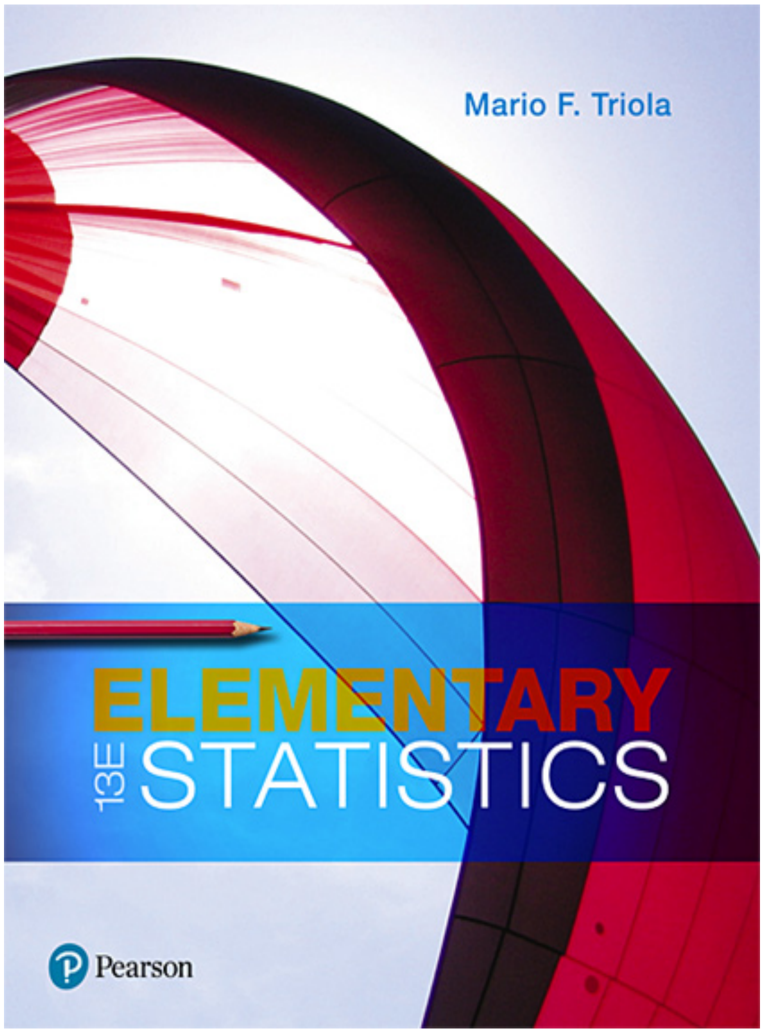
Elementary Statistics

Thirteenth Edition



Chapter 5

Probability Distributions

Probability Distributions

5-1 Probability Distributions

5-2 Binomial Probability Distributions

5-3 Poisson Probability Distributions

Key Concept

This section introduces the concept of a **random variable** and the concept of a **probability distribution.**

We illustrate how a **probability histogram** is a graph that visually depicts a probability distribution.

We show how to find the important parameters of mean, standard deviation, and variance for a probability distribution.

Most importantly, we describe how to determine whether outcomes are **significant** (significantly low or significantly high).

Basic Concepts of Probability Distribution (1 of 4)

- Random Variable
  - A **random variable** is a variable (typically represented by x) that has a single numerical value, determined by chance, for each outcome of a procedure.

Basic Concepts of Probability Distribution (2 of 4)

- Probability Distribution
  - A **probability distribution** is a description that gives the probability for each value of the random variable. It is often expressed in the format of a table, formula, or graph.

Basic Concepts of Probability Distribution (3 of 4)

- Discrete Random Variable
  - A **discrete random variable** has a collection of values that is finite or countable. (If there are infinitely many values, the number of values is countable if it is possible to count them individually, such as the number of tosses of a coin before getting heads.)

Basic Concepts of Probability Distribution (4 of 4)

- Continuous Random Variable
  - A **continuous random variable** has infinitely many values, and the collection of values is not countable. (That is, it is impossible to count the individual items because at least some of them are on a continuous scale, such as body temperatures.)

Probability Distribution Requirements (1 of 2)

Every probability distribution must satisfy each of the following three requirements.

1. There is a **numerical** (not categorical) random variable *x*, and its number values are associated with corresponding probabilities.
2. $\sum P(x) = 1$ where *x* assumes all possible values. (The sum of all probabilities must be 1, but sums such as 0.999 or 1.001 are acceptable because they result from rounding errors.)

Probability Distribution Requirements (2 of 2)

3. $0 \leq P(x) \leq 1$ for every individual value of the random variable x. (That is, each probability value must be between 0 and 1 inclusive.)

Example: Coin Toss (1 of 3)

Let's consider tossing two coins, with the following random variable:

x = number of heads when two coins are tossed

The above x is a random variable because its numerical values depend on chance.

***x*: Number of Heads When Two Coins Are Tossed**	***P*(*x*)**
0	0.25
1	0.50
2	0.25

Example: Coin Toss (2 of 3)

With two coins tossed, the number of heads can be 0, 1, or 2, and the table is a probability distribution because it gives the probability for each value of the random variable *x* and it satisfies the three requirements listed earlier:

1. The variable *x* is a **numerical** random variable, and its values are associated with probabilities.
2. $\sum P(x) = 0.25 + 0.50 + 0.25 = 1$
3. Each value of $P(x)$ is between 0 and 1.

Example: Coin Toss (3 of 3)

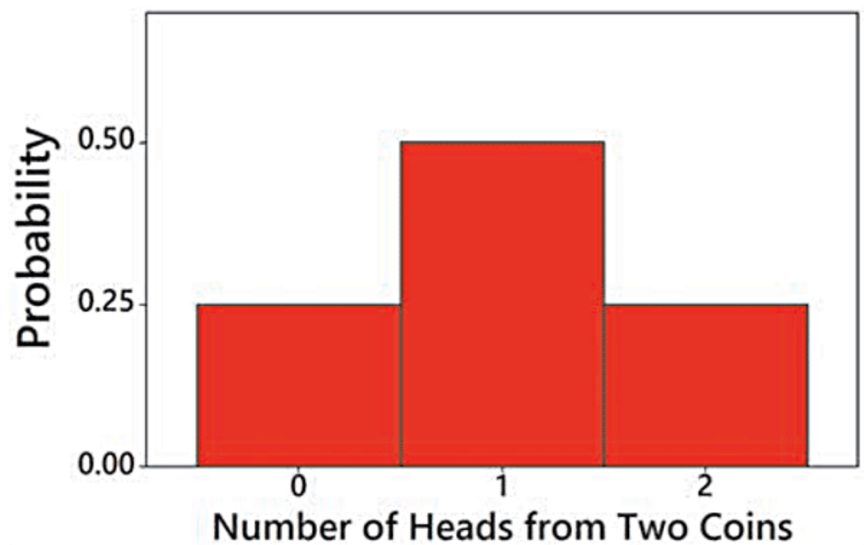
The random variable *x* in the table is a discrete random variable, because it has three possible values (0, 1, 2), and three is a finite number, so this satisfies the requirement of being finite.

***x*: Number of Heads When Two Coins Are Tossed**	***P*(*x*)**
0	0.25
1	0.50
2	0.25

Probability Histogram: Graph of a Probability Distribution

A probability histogram is similar to a relative frequency histogram, but the vertical scale shows **probabilities** instead of relative frequencies based on actual sample results.

Probability Histogram for Number of Heads When Two Coins Are Tossed



Probability Formula

A probability distribution could also be in the form of a formula. Consider the formula

$$P(x) = \frac{1}{2(2-x)!x!} \quad \text{(where } x \text{ can be 0, 1 or 2).}$$

We find that $P(0) = 0.25$, $P(1) = 0.50$, and $P(2) = 0.25$. The probabilities found using this formula are the same as those in the table.

Example: Job Interview Mistakes (1 of 2)

Hiring managers were asked to identify the biggest mistakes that job applicants make during an interview, and the table below is based on their responses (based on data from an Adecco survey). Does the table below describe a probability distribution?

x	***P*(*x*)**
Inappropriate attire	0.50
Being late	0.44
Lack of eye contact	0.33
Checking phone or texting	0.30
Total	**1.57**

Example: Job Interview Mistakes (2 of 2)

Solution

The table violates the first requirement because *x* is not a **numerical** random variable. The “values” of *x* are categorical data, not numbers.

The table also violates the second requirement because the sum of the probabilities is 1.57, but that sum should be 1.

Because the three requirements are not all satisfied, we conclude that the table does **not** describe a probability distribution.

Parameters of a Probability Distribution (1 of 3)

Remember that with a probability distribution, we have a description of a **population** instead of a sample, so the values of the mean, standard deviation, and variance are **parameters,** not statistics.

The mean, variance, and standard deviation of a discrete probability distribution can be found with the following formulas:

Parameters of a Probability Distribution (2 of 3)

- **Mean, *μ*,** for a probability distribution

$$\mu = \sum [x \cdot P(x)]$$

- **Variance**, ***σ*²**, for a probability distribution

$$\sigma^2 = \sum \left[(x - \mu)^2 \cdot P(x) \right]$$ (This format is easier to understand.)

- **Variance**, ***σ*²**, for a probability distribution

$$\sigma^2 = \sum \left[x^2 \cdot P(x) \right] - \mu^2$$ (This format is easier for manual calculations.)

Parameters of a Probability Distribution (3 of 3)

- **Standard deviation**, ***σ***, for a probability distribution

$$\sigma = \sqrt{\sum\left[x^2 \cdot P(x)\right] - \mu^2}$$

Expected Value (1 of 2)

- Expected Value
  - The **expected value** of a discrete random variable x is denoted by E, and it is the mean value of the outcomes, so $E = \mu$ and E can also be found by evaluating $\sum[x \cdot P(x)]$.

Example: Finding the Mean, Variance, and Standard Deviation

The table describes the probability distribution for the number of heads when two coins are tossed. Find the mean, variance, and standard deviation for the probability distribution described.

***x*: Number of Heads When Two Coins Are Tossed**	***P*(*x*)**
0	0.25
1	0.50
2	0.25

Example: Finding the Mean, Variance, and Standard Deviation (2 of 5)

x	$P(x)$	$x \cdot P(x)$	$(x-\mu)^2 \cdot P(x)$
0	0.25	$0 \cdot 0.25 = 0.00$	$(0-1.0)^2 \cdot 0.25 = 0.25$
1	0.50	$1 \cdot 0.50 = 0.50$	$(1-1.0)^2 \cdot 0.50 = 0.00$
2	0.25	$2 \cdot 0.25 = 0.50$	$(2-1.0)^2 \cdot 0.25 = 0.25$
Total		1.00 ↑ $\mu = \sum [x \cdot P(x)]$	0.50 ↑ $\sigma^2 = \sum [(x-\mu)^2 \cdot P(x)]$

Solution

The two columns at the left describe the probability distribution. The two columns at the right are for the purposes of the calculations required.

Example: Finding the Mean, Variance, and Standard Deviation (3 of 5)

x	$P(x)$	$x \cdot P(x)$	$(x-\mu)^2 \cdot P(x)$
0	0.25	$0 \cdot 0.25 = 0.00$	$(0 - 1.0)^2 \cdot 0.25 = 0.25$
1	0.50	$1 \cdot 0.50 = 0.50$	$(1 - 1.0)^2 \cdot 0.50 = 0.00$
2	0.25	$2 \cdot 0.25 = 0.50$	$(2 - 1.0)^2 \cdot 0.25 = 0.25$
Total		1.00 ↑ $\mu = \sum [x \cdot P(x)]$	0.50 ↑ $\sigma^2 = \sum [(x-\mu)^2 \cdot P(x)]$

Solution

Mean: $\sum [x \cdot P(x)] = 1.0$

Variance: $\sigma^2 = \sum \left[(x-\mu)^2 \cdot P(x) \right] = 0.5$

Example: Finding the Mean, Variance, and Standard Deviation (4 of 5)

x	$P(x)$	$x \cdot P(x)$	$(x - \mu)^2 \cdot P(x)$
0	0.25	$0 \cdot 0.25 = 0.00$	$(0 - 1.0)^2 \cdot 0.25 = 0.25$
1	0.50	$1 \cdot 0.50 = 0.50$	$(1 - 1.0)^2 \cdot 0.50 = 0.00$
2	0.25	$2 \cdot 0.25 = 0.50$	$(2 - 1.0)^2 \cdot 0.25 = 0.25$
Total		1.00 ↑ $\mu = \sum [x \cdot P(x)]$	0.50 ↑ $\sigma^2 = \sum [(x - \mu)^2 \cdot P(x)]$

Solution

The standard deviation is the square root of the variance, so

$$\text{Standard deviation: } \sigma = \sqrt{0.5}$$
$$= 0.707107 = 0.7$$

Example: Finding the Mean, Variance, and Standard Deviation

Interpretation

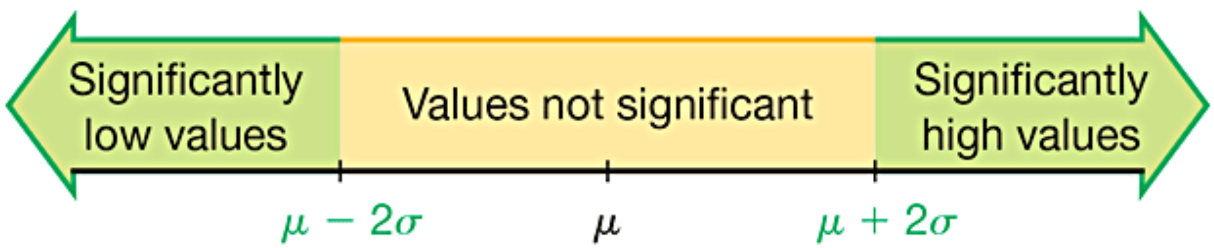
When tossing two coins, the mean number of heads is 1.0 head, the variance is 0.50 heads^2, and the standard deviation is 0.7 head.

Also, the expected value for the number of heads when two coins are tossed is 1.0 head, which is the same value as the mean. If we were to collect data on a large number of trials with two coins tossed in each trial, we expect to get a mean of 1.0 head.

Identifying Significant Results with the Range Rule of Thumb

Range Rule of Thumb for Identifying Significant Values

- **Significantly low** values are $(\mu - 2\sigma)$ or lower.
- **Significantly high** values are $(\mu + 2\sigma)$ or higher.
- **Values not significant**: Between $(\mu - 2\sigma)$ and $(\mu + 2\sigma)$.



Pearson

Example: Identifying Significant Results with the Range Rule of Thumb (1 of 3)

We found that when tossing two coins, the mean number of heads is $\mu = 1.0$ head and the standard deviation is $\sigma = 0.7$ head. Use those results and the range rule of thumb to determine whether 2 heads is a significantly high number of heads.

Example: Identifying Significant Results with the Range Rule of Thumb (2 of 3)

Solution

Using the range rule of thumb, the outcome of 2 heads is significantly high if it is greater than or equal to $(\mu + 2\sigma)$. With $\mu = 1.0$ head $\sigma = 0.7$ head, we get

$$(\mu + 2\sigma) = 1 + 2(0.7) = 2.4 \text{ heads}$$

Significantly high numbers of heads are 2.4 and above.

Example: Identifying Significant Results with the Range Rule of Thumb (3 of 3)

Interpretation

Based on these results, we conclude that 2 heads is not a significantly high number of heads (because 2 is not greater than or equal to 2.4).

Identifying Significant Results with Probabilities: (1 of 2)

- Significantly high number of successes:
  - *x* successes among *n* trials is a **significantly high** number of successes if the probability of *x* or more successes is 0.05 or less. That is, *x* is a significantly high number of successes if *P*(*x* **or more**) ≤ 0.05.

The value 0.05 is not absolutely rigid. Other values, such as 0.01, could be used to distinguish between results that are significant and those that are not significant.

Identifying Significant Results with Probabilities: (2 of 2)

- Significantly low number of successes:
  - *x* successes among *n* trials is a **significantly low** number of successes if the probability of *x* or fewer successes is 0.05 or less. That is, *x* is a significantly low number of successes if *P(x* **or fewer***)* ≤ 0.05*.*

The value 0.05 is not absolutely rigid. Other values, such as 0.01, could be used to distinguish between results that are significant and those that are not significant.

The Rare Event Rule for Inferential Statistics

If, under a given assumption, the probability of a particular outcome is very small and the outcome occurs **significantly less than or significantly greater than** what we expect with that assumption, we conclude that the assumption is probably not correct.

Expected Value (2 of 2)

The expected value of a random variable x is equal to the mean μ. We can therefore find the expected value by computing $\sum[x \cdot P(x)]$, just as we do for finding the value of μ.

Example: Be a Better Bettor (1 of 6)

You have $5 to place on a bet in the Golden Nugget casino in Las Vegas. You have narrowed your choice to one of two bets:

Roulette: Bet on the number 7 in roulette.

Craps: Bet on the “pass line” in the dice game of craps.

Example: Be a Better Bettor (2 of 6)

a. If you bet \$5 on the number 7 in roulette, the probability of losing \$5 is $\frac{37}{38}$ and the probability of making a net gain of \$175 is $\frac{1}{38}$. (The prize is \$180, including your \$5 bet, so the net gain is \$175.) Find your expected value if you bet \$5 on the number 7 in roulette.

Example: Be a Better Bettor (3 of 6)

b. If you bet $5 on the pass line in the dice game of craps, the probability of losing $5 is $\frac{251}{495}$ and the probability of making a net gain of $5 is $\frac{244}{495}$. (If you bet $5 on the pass line and win, you are given $10 that includes your bet, so the net gain is $5.) Find your expected value if you bet $5 on the pass line.

Which of the preceding two bets is better in the sense of producing higher expected value?

Example: Be a Better Bettor (4 of 6)

Solution

a. **Roulette** The probabilities and payoffs for betting $5 on the number 7 in roulette are summarized in the table. The table also shows that the expected value is $\sum[x \cdot P(x)] = -26¢$. That is, for every $5 bet on the number 7, you can expect to **lose** an average of 26¢.

Event	***x***	***P*(*x*)**	***x* · *P*(*x*)**
Lose	–$5	$\frac{37}{38}$	–$4.868421
Win (net gain)	$175	$\frac{1}{38}$	$4.605263
Total			–$0.26 (rounded) (or -26¢)

Example: Be a Better Bettor (5 of 6)

Solution

b. **Craps Game** The probabilities and payoffs for betting $5 on the pass line in craps are summarized in the table. The table also shows that the expected value is $\sum[x \cdot P(x)] = -7¢$. That is, for every $5 bet on the pass line, you can expect to lose an average of 7¢.

Event	***x***	***P*(*x*)**	***x* · *P*(*x*)**
Lose	–$5	$\frac{251}{495}$	–$2.535353
Win (net gain)	$5	$\frac{244}{495}$	–$2.464646
Total			–$0.07 (rounded) (or –7¢)

Example: Be a Better Bettor (6 of 6)

Interpretation

The \$5 bet in roulette results in an expected value of −26¢ and the \$5 bet in craps results in an expected value of −7¢. Because you are better off losing 7¢ instead of losing 26¢, the craps game is better in the long run, even though the roulette game provides an opportunity for a larger payoff when playing the game once.