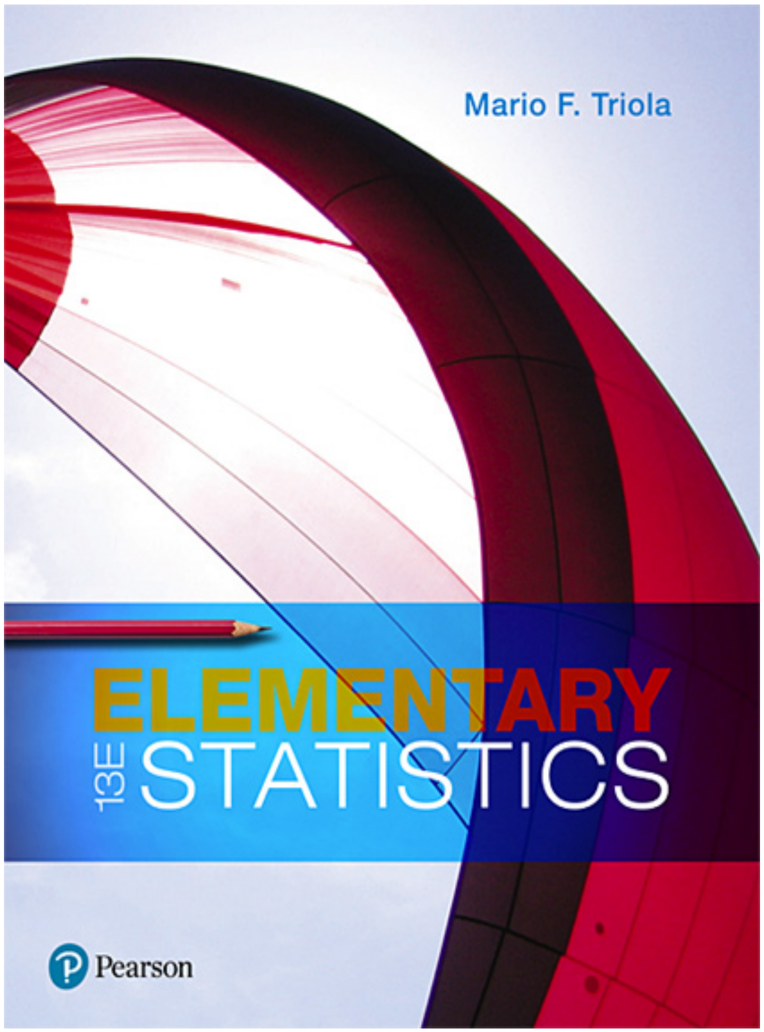
Elementary Statistics

Thirteenth Edition



Chapter 5

Probability Distributions

Probability Distributions

5-1 Probability Distributions

5-2 Binomial Probability Distributions

5-3 Poisson Probability Distributions

Key Concept

In this section, we introduce **Poisson probability distributions,** which are another category of discrete probability distributions.

Poisson Probability Distribution (1 of 2)

- Poisson Probability Distribution
  - A **Poisson probability distribution** is a discrete probability distribution that applies to occurrences of some event **over a specified interval.** The random variable x is the number of occurrences of the event in an interval. The interval can be time, distance, area, volume, or some similar unit.

Poisson Probability Distribution (2 of 2)

The probability of the event occurring *x* times over an interval is given by

$$P(x) = \frac{\mu^x \cdot e^{-\mu}}{x!}$$

where $e \approx 2.71828$, μ = mean number of occurrences of the event in the intervals

Requirements for the Poisson Probability Distribution

1. The random variable x is the number of occurrences of an event **in some interval.**
2. The occurrences must be **random.**
3. The occurrences must be **independent** of each other.
4. The occurrences must be **uniformly distributed** over the interval being used.

Poisson Probability Distribution Parameters and Properties

Parameters of the Poisson Probability Distribution

- The mean is μ.
- The standard deviation is $\sigma = \sqrt{\mu}$.

Properties of the Poisson Probability Distribution

- A particular Poisson distribution is determined only by the mean, μ.
- A Poisson distribution has possible x values of 0, 1, 2, . . . with no upper limit.

Example: Atlantic Hurricanes (1 of 5)

For the 55-year period since 1960, there were 336 Atlantic hurricanes. Assume the Poisson distribution.

a. Find μ, the mean number of hurricanes per year.

b. Find the probability that in a randomly selected year, there are exactly 8 hurricanes. Find $P(8)$, where $P(x)$ is the probability of x Atlantic hurricanes in a year.

Example: Atlantic Hurricanes (2 of 5)

For the 55-year period since 1960, there were 336 Atlantic hurricanes. Assume the Poisson distribution.

c. In this 55-year period, there were actually 5 years with 8 Atlantic hurricanes. How does this actual result compare to the probability found in part (b)? Does the Poisson distribution appear to be a good model in this case?

Example: Atlantic Hurricanes (3 of 5)

Solution

a. The Poisson distribution applies because we are dealing with the occurrences of an event (hurricanes) over some interval (a year). The mean number of hurricanes per year is

$$\mu = \frac{\text{number of hurricanes}}{\text{number of years}} = \frac{336}{55} = 6.1$$

Example: Atlantic Hurricanes (4 of 5)

Solution

b. The probability of x = 8 hurricanes in a year is as follows (with x = 8, μ = 6.1, and e = 2.71828):

$$P(8) = \frac{\mu^x \cdot e^{-\mu}}{x!} = \frac{6.1^8 \cdot 2.71828^{-6.1}}{8!}$$

$$= \frac{(1{,}917{,}073.13)(0.0022428769)}{40{,}320}$$

$$= 0.107$$

The probability of exactly 8 hurricanes in a year is $P(8) = 0.107$.

Example: Atlantic Hurricanes (5 of 5)

Solution

c. The probability of $P(8) = 0.107$ from part (b) is the likelihood of getting 8 Atlantic hurricanes in 1 year. In 55 years, the expected number of years with 8 Atlantic hurricanes is $55 \times 0.107 = 5.9$ years. The expected number of years with 8 hurricanes is 5.9, which is reasonably close to the 5 years that actually had 8 hurricanes, so in this case, the Poisson model appears to work quite well.

Poisson Distribution as Approximation to Binomial (1 of 2)

The Poisson distribution is sometimes used to approximate the binomial distribution when n is large and p is small. One rule of thumb is to use such an approximation when the following two requirements are both satisfied.

1. $n \geq 100$
2. $np \leq 10$

Poisson Distribution as Approximation to Binomial (2 of 2)

If both requirements are satisfied and we want to use the Poisson distribution as an approximation to the binomial distribution, we need a value for μ.

Mean for Poisson as an Approximation to Binomial

$$\mu = np$$

Example: Maine Pick 4 (1 of 4)

In the Maine Pick 4 game, you pay 50¢ to select a sequence of four digits (0–9), such as 1377. If you play this game once every day, find the probability of winning at least once in a year with 365 days.

Example: Maine Pick 4 (2 of 4)

Solution

The time interval is a day, and playing once each day results in $n = 365$ games. There is one winning set of numbers among the 10,000 that are possible (from 0000 to 9999). The probability of a win is $p = \frac{1}{10{,}000}$. With $n = 365$ and $p = \frac{1}{10{,}000}$, the conditions $n \geq 100$ and $np \leq 10$ are both satisfied, so we can use the Poisson distribution as an approximation to the binomial distribution. We first need the value of μ, which is found as follows:

$$\mu = np = 365 \cdot \frac{1}{10{,}000} = 0.0365$$

Example: Maine Pick 4 (3 of 4)

Solution

Having found the value of μ, we can proceed to find the probability for specific values of x. We want the probability that x is "at least 1,". We will use the strategy of first finding $P(0)$, the probability of no wins in 365 days. The probability of at least one win can then be found by subtracting that result from 1. We find $P(0)$ by using $x = 0$, $\mu = 0.0365$, and $e = 2.71828$, as shown here:

$$P(0) = \frac{\mu^x \cdot e^{-\mu}}{x!} = \frac{0.0365^0 \; 2.71828^{-0.0365}}{0!} = 0.9642$$

Example: Maine Pick 4 (4 of 4)

Solution

Using the Poisson distribution as an approximation to the binomial distribution, we find that there is a 0.9642 probability of no wins, so the probability of at least one win is $1 - 0.9642 = 0.0358$. If we use the binomial distribution, we get a probability of 0.0358, so the Poisson distribution works quite well here.