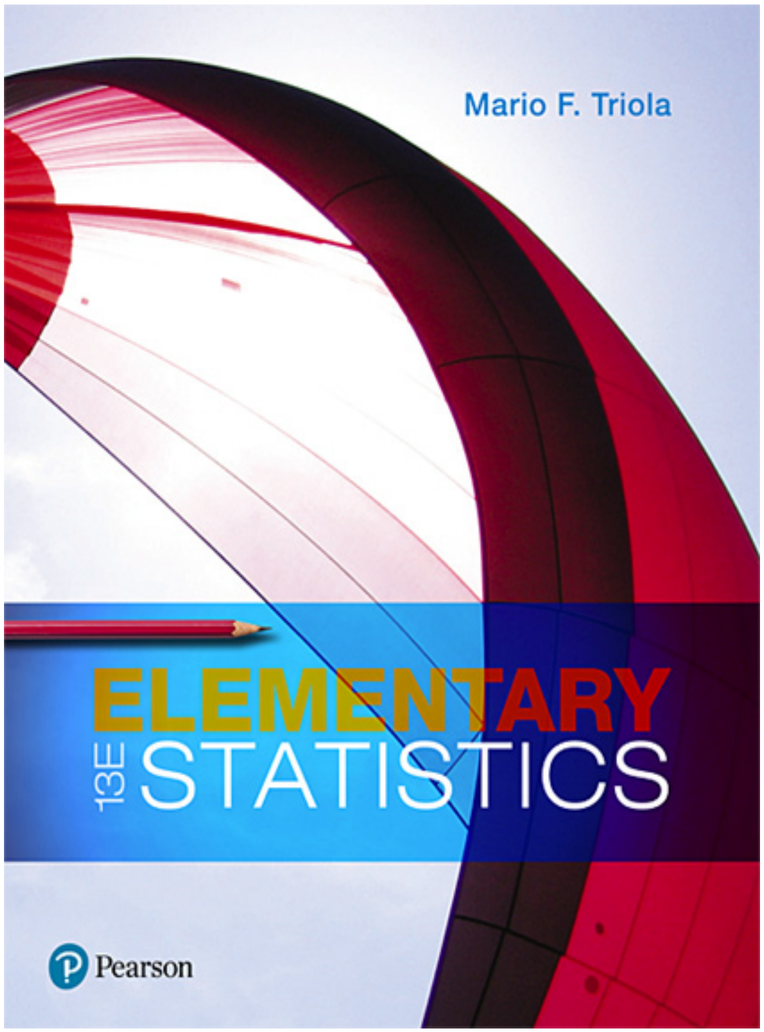
Elementary Statistics

Thirteenth Edition



Chapter 9

Inferences from Two Samples

Inferences from Two Samples

9-1 Two Proportions

9-2 Two Means: Independent Samples

9-3 Two Dependent Samples (Matched Pairs)

9-4 Two Variances or Standard Deviations

Key Concept

In this section we present methods for (1) testing a claim made about two population proportions and (2) constructing a confidence interval estimate of the difference between two population proportions. The methods of this chapter can also be used with probabilities or the decimal equivalents of percentages.

Inferences About Two Proportions: Objectives

Objectives

1. **Hypothesis Test:** Conduct a hypothesis test of a claim about two population proportions.
2. **Confidence Interval:** Construct a confidence interval estimate of the difference between two population proportions.

Inferences About Two Proportions: Notation for Two Proportions

For population 1 we let

p_1 = **population** proportion

$\hat{p}_1 = \frac{x_1}{n_1}$ (**sample** proportion)

n_1 = size of the first sample

$\hat{q}_1 = 1 - \hat{p}_1$ (complement of $\hat{p}_1$)

x_1 = number of successes in the first sample

The corresponding notations p_2, n_2, x_2, $\hat{p}_2$, and $\hat{q}_2$ apply to population 2.

Inferences About Two Proportions: Pooled Sample Proportion

The **pooled sample proportion** is denoted by $\bar{p}$ and it combines the two sample proportions into one proportion, as shown here:

$$\bar{p} = \frac{x_1 + x_2}{n_1 + n_2}$$

$$\bar{q} = 1 - \bar{p}$$

Inferences About Two Proportions: Requirements

1. The sample proportions are from two simple random samples.
2. The two samples are **independent.** (Samples are **independent** if the sample values selected from one population are not related to or somehow naturally paired or matched with the sample values from the other population.)
3. For each of the two samples, there are at least 5 successes and at least 5 failures. (That is, $n\hat{p} \geq 5$ and $n\hat{q} \geq 5$ for each of the two samples).

Inferences About Two Proportions: Test Statistic for Two Proportions (with H_0: $p_1 = p_2$)

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\frac{\bar{p}\bar{q}}{n_1} + \frac{\bar{p}\bar{q}}{n_2}}}$$

where $p_1 - p_2 = 0$ (assumed in the null hypothesis)

Where $\bar{p} = \frac{x_1 + x_2}{n_1 + n_2}$ (**pooled** sample proportion) and $\bar{q} = 1 - \bar{p}$

P-Value and Critical Values

- ***P*-Value:** *P*-values are automatically provided by technology. If technology is not available, use Table A-2 (standard normal distribution) and find the *P*-value using the procedure given in Figure 8-3 on page 364.
- **Critical Values:** Use Table A-2. (Based on the significance level α, find critical values by using the same procedures introduced in Section 8-1.)

Confidence Interval Estimate of $p_1 - p_2 = 0$

The confidence interval estimate of the difference $p_1 - p_2$ is

$$(\hat{p}_1 - \hat{p}_2) - E < (p_1 - p_2) < (\hat{p}_1 - \hat{p}_2) + E$$

where the margin of error E **is given by**

$$E = z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}}.$$

Equivalent Methods

When testing a claim about two population proportions:

- The *P*-value method and the critical value method are equivalent.
- The confidence interval method is **not** equivalent to the *P*-value method or the critical value method.

Hypothesis Tests

For tests of hypotheses made about two population proportions, we consider only tests having a null hypothesis of $p_1 = p_2$ (so the null hypothesis is $H_0 : p_1 = p_2$).

With the assumption that $p_1 = p_2$, the estimates of $\hat{p}_1$ and $\hat{p}_2$ are combined to provide the best estimate of the common value of $\hat{p}_1$ and $\hat{p}_2$, and that combined value is the pooled sample proportion $\bar{p}$ given in the preceding slides.

Example: Proportions of Cars with Rear License Plates Only: Are the Proportions the Same in Connecticut and New York? (1 of 14)

Connecticut and New York are contiguous states, both having laws that require front and rear license plates. The proportion of Connecticut "illegal" cars with rear license plates only is $\frac{239}{2049}$, or 11.7%. The proportion of New York "illegal" cars with rear license plates only is $\frac{9}{550}$, or 1.6%. The sample percentages of 11.7% and 1.6% are obviously different, but are they **significantly** different?

Example: Proportions of Cars with Rear License Plates Only: Are the Proportions the Same in Connecticut and New York? (2 of 14)

	Connecticut	New York
Cars with rear license plate only	239	9
Cars with front and rear license plates	1810	541
Total	**2049**	**550**

$$\text{Connecticut: } \hat{p}_1 = \frac{239}{2049} = 0.117$$

$$\text{New York: } \hat{p}_2 = \frac{9}{550} = 0.016$$

Use a 0.05 significance level and the *P*-value method to test the claim that Connecticut and New York have the same proportion of cars with rear license plates only.

Example: Proportions of Cars with Rear License Plates Only: Are the Proportions the Same in Connecticut and New York? (3 of 14)

Solution

Requirement Check (1) The two samples are simple random samples (trust the author!). (2) The two samples are independent because cars in the samples are not matched or paired in any way. (3) Let’s consider a “success” to be a car with a rear license plate only. For Connecticut, the number of successes is 239 and the number of failures (cars with front and rear license plates) is 1810, so they are both at least 5. For New York, there are 9 successes and 541 failures, and they are both at least 5. The requirements are satisfied.

Example: Proportions of Cars with Rear License Plates Only: Are the Proportions the Same in Connecticut and New York? (4 of 14)

Solution

Step 1: The claim that "Connecticut and New York have the same proportion of cars with rear license plates only" can be expressed as $p_1 = p_2$.

Step 2: If $p_1 = p_2$ is false, then $p_1 \neq p_2$.

Step 3: Because the claim of $p_1 \neq p_2$ does not contain equality, it becomes the alternative hypothesis. The null hypothesis is the statement of equality, so we have

$$H_0\colon p_1 = p_2 \qquad H_1\colon p_1 \neq p_2$$

Example: Proportions of Cars with Rear License Plates Only: Are the Proportions the Same in Connecticut and New York? (5 of 14)

Solution

Step 4: The significance level was specified as $\alpha = 0.05$, so we use $\alpha = 0.05$.

Step 5: This step and the following step can be circumvented by using technology; see the display that follows this example. If not using technology, we use the normal distribution as an approximation to the binomial distribution.

Example: Proportions of Cars with Rear License Plates Only: Are the Proportions the Same in Connecticut and New York? (6 of 14)

Solution

Step 5 (con't): We estimate the common value of p_1 and p_2 with the pooled sample estimate p calculated as shown below, with extra decimal places used to minimize rounding errors in later calculations.

$$\bar{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{239 + 9}{2049 + 550} = 0.09542132$$

$$\bar{q} = 1 - \bar{p} = 1 - 0.09542132 = 0.90457868.$$

Example: Proportions of Cars with Rear License Plates Only: Are the Proportions the Same in Connecticut and New York? (7 of 14)

Solution

Step 6: Because we assume in the null hypothesis that $p_1 = p_2$, the value of $p_1 - p_2$ is 0 in the following calculation of the test statistic:

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\frac{\bar{p}\bar{q}}{n_1} + \frac{\bar{p}\bar{q}}{n_2}}}$$

$$= \frac{\left(\frac{239}{2049} - \frac{9}{550}\right) - 0}{\sqrt{\frac{(0.09542132)(0.90457868)}{2049} + \frac{(0.09542132)(0.90457868)}{550}}}$$

$$= 7.11$$

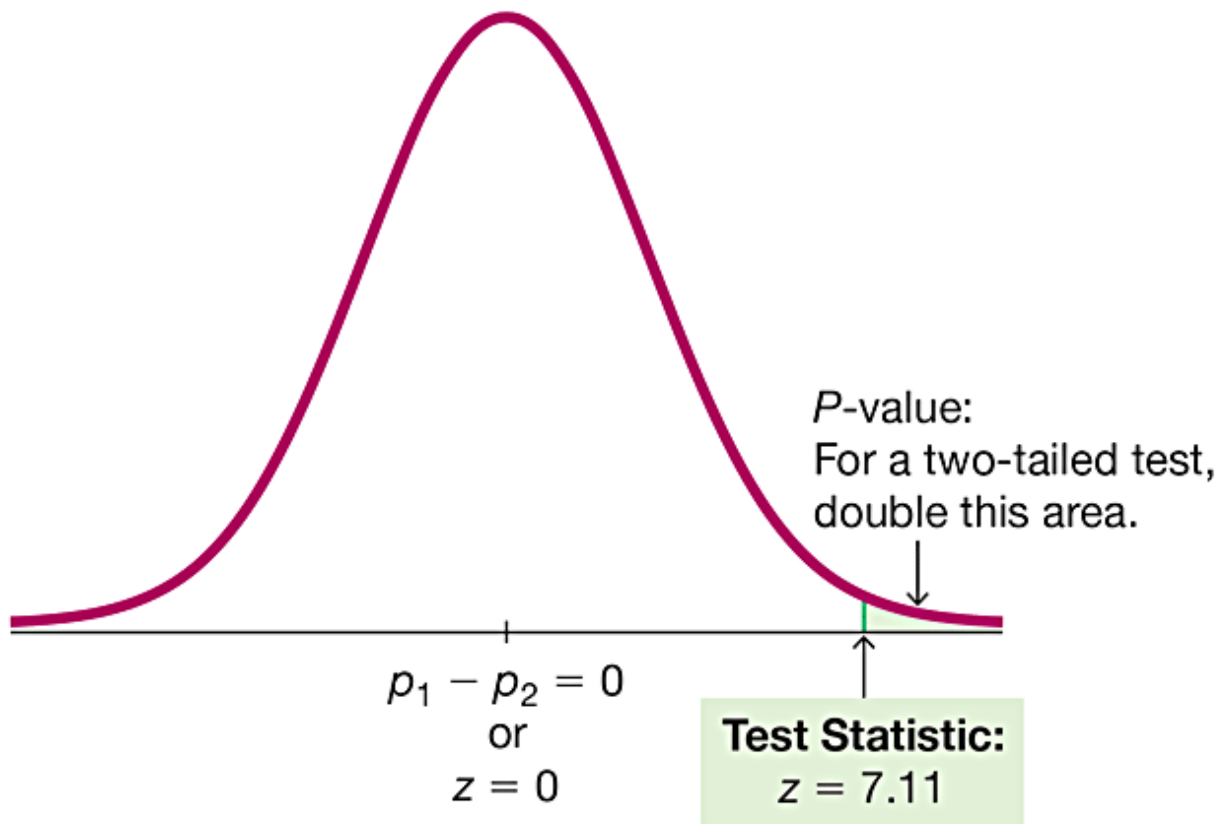
Example: Proportions of Cars with Rear License Plates Only: Are the Proportions the Same in Connecticut and New York? (8 of 14)

Solution

Step 6 (con't): This is a two-tailed test, so the *P*-value is twice the area to the right of the test statistic $z = 7.11$. Refer to Table A-2 and find that the area to the right of the test statistic $z = 7.11$ is 0.0001, so the *P*-value is 0.0002. Technology provides a more accurate *P*-value of 0.00000000000119, which is often expressed as 0.0000 or "*P*-value < 0.0001."

Example: Proportions of Cars with Rear License Plates Only: Are the Proportions the Same in Connecticut and New York? (9 of 14)

Solution



Example: Proportions of Cars with Rear License Plates Only: Are the Proportions the Same in Connecticut and New York? (10 of 14)

Solution

Step 7: Because the *P*-value of 0.0000 is less than the significance level of $\alpha = 0.05$, we reject the null hypothesis of $p_1 = p_2$.

Example: Proportions of Cars with Rear License Plates Only: Are the Proportions the Same in Connecticut and New York? (11 of 14)

Interpretation

We must address the original claim that “Connecticut and New York have the same proportion of cars with rear license plates only.” Because we reject the null hypothesis, we conclude that there is sufficient evidence to warrant rejection of the claim that $p_1 = p_2$. That is, there is sufficient evidence to conclude that Connecticut and New York have **different** proportions of cars with rear license plates only. It’s reasonable to speculate that enforcement of the license plate laws is much stricter in New York than in Connecticut, and that is why Connecticut car owners are less likely to install the front license plate.

Example: Proportions of Cars with Rear License Plates Only: Are the Proportions the Same in Connecticut and New York? (12 of 14)

Technology

Software and calculators usually provide a *P*-value, so the *P*-value method is typically used for testing a claim about two proportions. See the Statdisk results showing the test statistic of $z = 7.11$(rounded) and the *P*-value of 0.0000.

Statdisk

```
Pooled proportion: 0.0954213
Test Statistic, z: 7.1074
Critical z: ±1.9600
P-Value: 0.0000

95% Confidence interval:
0.0827974 < p1-p2 < 0.1177599
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Example: Proportions of Cars with Rear License Plates Only: Are the Proportions the Same in Connecticut and New York? (13 of 14)

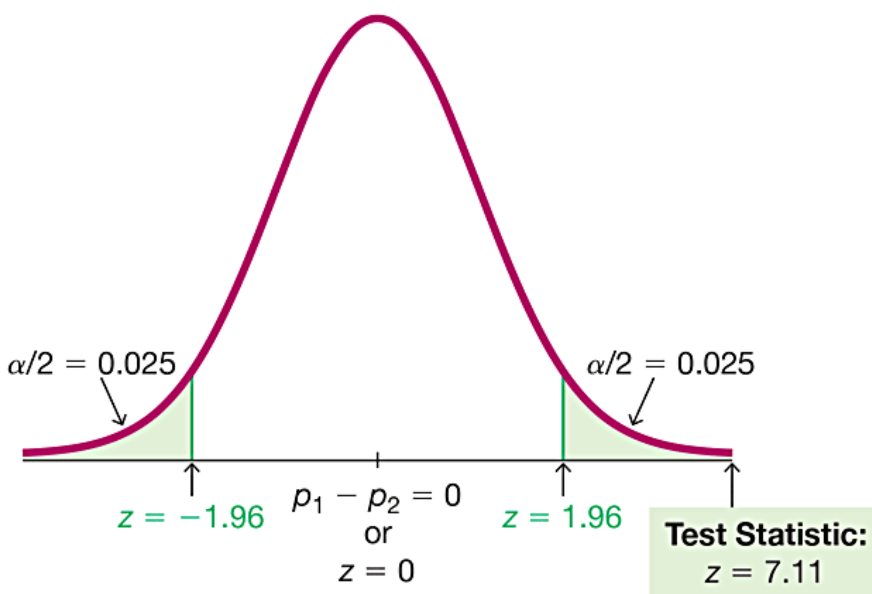
Solution (Critical Value Method)

The critical value method of testing hypotheses can also be used. In Step 6, find the critical values. With a significance level of $\alpha = 0.05$ in a two-tailed test based on the normal distribution, we refer to Table A-2 and find that an area of $\alpha = 0.05$ divided equally between the two tails corresponds to the critical values of $z = \pm 1.96$.

Example: Proportions of Cars with Rear License Plates Only: Are the Proportions the Same in Connecticut and New York? (14 of 14)

Solution (Critical Value Method)

We can see that the test statistic of $z = 7.11$ falls within the critical region beyond the critical value of 1.96. We again reject the null hypothesis. The conclusions are the same.



Example: Confidence Interval for Claim About Two Proportions (1 of 5)

Use the same sample data given to construct a 95% confidence interval estimate of the difference between the two population proportions. What does the result suggest about the claim that “Connecticut and New York have the same proportion of cars with rear license plates only”?

Example: Confidence Interval for Claim About Two Proportions (2 of 5)

Solution

Requirement Check We are using the same data and the same requirement check applies here. The confidence interval can be found using technology.

Statdisk

Pooled proportion: 0.0954213
Test Statistic, z: 7.1074
Critical z: ±1.9600
P-Value: 0.0000

95% Confidence interval:
0.0827974 < p1-p2 < 0.1177599

Example: Confidence Interval for Claim About Two Proportions (3 of 5)

Solution

If not using technology, proceed as follows.

With a 95% confidence level, $z_{\frac{\alpha}{2}} = 1.96$. We calculate the value of the margin of error E as shown here.

$$E = z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}_1 \hat{q}_1}{n_1} + \frac{\hat{p}_2 \hat{q}_2}{n_2}}$$

$$= 1.96 \sqrt{\frac{\left(\frac{239}{2049}\right)\left(\frac{1810}{2049}\right)}{2049} + \frac{\left(\frac{9}{550}\right)\left(\frac{541}{550}\right)}{550}} = 0.017482$$

Example: Confidence Interval for Claim About Two Proportions (4 of 5)

Solution

With $\hat{p}_1 = \dfrac{239}{2049} = 0.116642$ and $\hat{p}_2 = \dfrac{9}{550} = 0.016364$, we get $\hat{p}_1 - \hat{p}_2 = 0.100278$. With $E = 0.017482$, the confidence interval is evaluated as follows, with the confidence interval limits rounded to three significant digits:

$$(\hat{p}_1 - \hat{p}_2) - E < (p_1 - p_2) < (\hat{p}_1 - \hat{p}_2) + E$$

$$0.100278 - 0.017482 < (p_1 - p_2) < 0.100278 + 0.017482$$

$$0.0828 < (p_1 - p_2) < 0.118$$

See the preceding Statdisk display showing the same confidence interval obtained here.

Example: Confidence Interval for Claim About Two Proportions (5 of 5)

Interpretation

The confidence interval limits do not contain 0, suggesting that there is a significant difference between the two proportions. The confidence interval suggests that the value of p_1 is greater than the value of p_2, so there does appear to be sufficient evidence to warrant rejection of the claim that "Connecticut and New York have the same proportion of cars with rear license plates only."

What Can We Do When Requirements Are Not Satisfied?

- Bad Samples are bad samples.
- If we violate the requirement that each of the two samples has at least 5 successes and at least 5 failures in a hypothesis test, we can use **Fisher's exact test.**
- If we violate the requirement that each of the two samples has at least 5 successes and at least 5 failures in a confidence interval, we can use bootstrap resampling methods.

Why Do the Procedures of This Section Work? (1 of 4)

The distribution of $\hat{p}_1$ can be approximated by a normal distribution with mean p_1, and standard deviation $\sqrt{\frac{p_1 q_1}{n_1}}$ and variance $\frac{p_1 q_1}{n_1}$.

The difference $\hat{p}_1 - \hat{p}_2$ will also be approximated by a normal distribution with mean $p_1 - p_2$ and variance

$$\sigma^2_{(\hat{p}_1 - \hat{p}_2)} = \sigma^2_{\hat{p}_1} + \sigma^2_{\hat{p}_2} = \frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}$$

The variance of the **differences** between two independent random variables is the **sum** of their individual variances.

Why Do the Procedures of This Section Work? (2 of 4)

The preceding variance leads to

$$\sigma_{(\hat{p}_1 - \hat{p}_2)} = \sqrt{\frac{\bar{p}\bar{q}}{n_1} + \frac{\bar{p}\bar{q}}{n_2}}$$

We now know that the distribution of $\hat{p}_1 - \hat{p}_2$ is approximately normal, with mean $p_1 - p_2$ and standard deviation as shown above, so the z test statistic has the form given earlier.

Why Do the Procedures of This Section Work? (3 of 4)

When constructing the confidence interval estimate of the difference between two proportions, we don’t assume that the two proportions are equal, and we estimate the standard deviation as

$$\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1}+\frac{\hat{p}_2\hat{q}_2}{n_2}}$$

Why Do the Procedures of This Section Work? (4 of 4)

In the test statistic

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\dfrac{\bar{p}\bar{q}}{n_1} + \dfrac{\bar{p}\bar{q}}{n_2}}}$$

use the positive and negative values of z (for two tails) and solve for $p_1 - p_2$.

The results are the limits of the confidence interval given earlier.