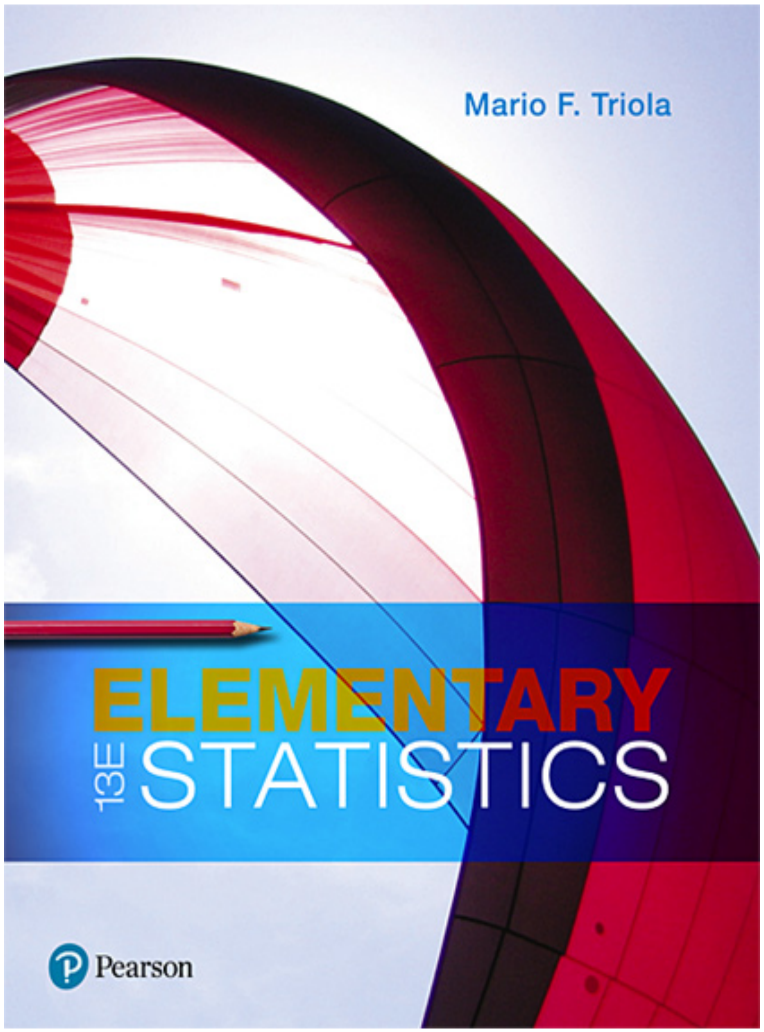
Elementary Statistics

Thirteenth Edition



Chapter 9

Inferences from Two Samples

Inferences from Two Samples

9-1 Two Proportions

9-2 Two Means: Independent Samples

9-3 Two Dependent Samples (Matched Pairs)

9-4 Two Variances or Standard Deviations

Key Concept

This section presents methods for using sample data from two independent samples to test hypotheses made about two population means or to construct confidence interval estimates of the difference between two population means.

Independent (1 of 2)

- Independent
  - Two samples are **independent** if the sample values from one population are not related to or somehow naturally paired or matched with the sample values from the other population.

Independent (2 of 2)

- Dependent
  - Two samples are **dependent** (or consist of **matched pairs**) if the sample values are somehow matched, where the matching is based on some inherent relationship.

Inferences About Two Means: Independent Samples: Objectives

Objectives

1. **Hypothesis Test:** Conduct a hypothesis test of a claim about two independent population means.
2. **Confidence Interval:** Construct a confidence interval estimate of the difference between two independent population means.

Inferences About Two Means: Independent Samples: Notation

Notation

For population 1 we let

μ_1 = **population** mean

$\bar{x}_1$ = **sample** mean

σ_1 = **population** standard deviation

s_1 = **sample** standard deviation

n_1 = size of the first sample

The corresponding notations μ_2, s_2, $\bar{x}_2$, s_2 , and n_2, apply to population 2.

Inferences About Two Means: Independent Samples: Requirements

Requirements

1. The values of σ_1 and σ_2 are unknown and we do not assume that they are equal.
2. The two samples are **independent.**
3. Both samples are **simple random samples**.
4. Either or both of these conditions are satisfied: The two sample sizes are both **large** (with $n_1 > 30$ and $n_2 > 30$) or both samples come from populations having normal distributions.

Inferences About Two Means: Independent Samples: Hypothesis Test Statistic for Two Means: Independent Samples (with H_0: $\mu_1 = \mu_2$) (1 of 3)

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

(where $\mu_1 - \mu_2$ is often assumed to be 0)

Pearson

Inferences About Two Means: Independent Samples: Hypothesis Test Statistic for Two Means: Independent Samples (with H_0: $\mu_1 = \mu_2$) (2 of 3)

Degrees of freedom

1. Use this simple and conservative estimate:
   df = smaller of $n_1 - 1$ and $n_2 - 1$.
2. Technologies typically use the more accurate but more difficult estimate given by the following formula:

$$\text{df} = \frac{(A+B)^2}{\dfrac{A^2}{n_1-1} + \dfrac{B^2}{n_2-1}} \quad \text{where } A = \frac{s_1^2}{n_1} \text{ and } B = \frac{s_2^2}{n_2}$$

Inferences About Two Means: Independent Samples: Hypothesis Test Statistic for Two Means: Independent Samples (with H_0: $\mu_1 = \mu_2$) (3 of 3)

***P*-Values:** *P*-values are automatically provided by technology. If technology is not available, refer to the *t* distribution in Table A-3. **Critical Values:** Refer to the *t* distribution in Table A-3.

Inferences About Two Means: Independent Samples: Confidence Interval Estimate of $\mu_1 - \mu_2$: Independent Samples

The confidence interval estimate of the difference $\mu_1 - \mu_2$ is

$$(\bar{x}_1 - \bar{x}_2) - E < (\mu_1 - \mu_2) < (\bar{x}_1 - \bar{x}_2) + E$$

where the margin of error E **is given by**

$$E = t_{\frac{\alpha}{2}} \sqrt{\frac{s_1^{\,2}}{n_1} + \frac{s_2^{\,2}}{n_2}}.$$

Equivalent Methods

The *P*-value method of hypothesis testing, the critical value method of hypothesis testing, and confidence intervals all use the same distribution and standard error, so they are all equivalent in the sense that they result in the same conclusions.

Example: Are Male Professors and Female Professors Rated Differently by Students? (1 of 10)

Listed below are student course evaluation scores for courses taught by female professors and male professors. Use a 0.05 significance level to test the claim that the two samples are from populations with the same mean. Does there appear to be a difference in evaluation scores of courses taught by female professors and male professors?

Female	4.3	4.3	4.4	4.0	3.4	4.7	2.9	4.0	4.3	3.4	3.4	3.3			
Male	4.5	3.7	4.2	3.9	3.1	4.0	3.8	3.4	4.5	3.8	4.3	4.4	4.1	4.2	4.0

Example: Are Male Professors and Female Professors Rated Differently by Students? (2 of 10)

Solution

Requirement Check (1) The values of the two population standard deviations are not known and we are not making an assumption that they are equal. (2) The two samples are independent because the female professors and male professors are not matched or paired in any way.

Example: Are Male Professors and Female Professors Rated Differently by Students? (3 of 10)

Solution

Requirement Check (3) The samples are simple random samples. (4) Both samples are small (30 or fewer), so we need to determine whether both samples come from populations having normal distributions. Normal quantile plots of the two samples suggest that the samples are from populations having distributions that are not far from normal.

The requirements are all satisfied.

Example: Are Male Professors and Female Professors Rated Differently by Students? (4 of 10)

Solution

Step 1: The claim that "the two samples are from populations with the same mean" can be expressed as $\mu_1 = \mu_2$.

Step 2: If the original claim is false, then $\mu_1 \neq \mu_2$.

Step 3: The alternative hypothesis is the expression not containing equality, and the null hypothesis is an expression of equality, so we have

$$H_0: \mu_1 = \mu_2 \quad H_1: \mu_1 \neq \mu_2$$

Example: Are Male Professors and Female Professors Rated Differently by Students? (5 of 10)

Solution

We now proceed with the assumption that $\mu_1 = \mu_2$, or $\mu_1 - \mu_2 = 0$.

Step 4: The significance level is $\alpha = 0.05$.

Step 5: Because we have two independent samples and we are testing a claim about the two population means, we use a *t* distribution with the test statistic given earlier in this section.

Example: Are Male Professors and Female Professors Rated Differently by Students? (6 of 10)

Solution

Step 6: The test statistic is calculated using these statistics (with extra decimal places) obtained from the listed sample data:

Females: $n = 12$, $\bar{x} = 3.866667$, $s = 0.563001$; males: $n = 15$, $\bar{x} = 3.993333$, $s = 0.395450$.

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{(3.866667 - 3.993333) - 0}{\sqrt{\frac{0.563001^2}{12} + \frac{0.395450^2}{15}}} = -0.660$$

Example: Are Male Professors and Female Professors Rated Differently by Students? (7 of 10)

Solution

***P*-Value** With test statistic $t = -0.660$, we refer to Table A-3 (t Distribution). The number of degrees of freedom is the smaller of $n_1 - 1$ and $n_2 - 1$, or the smaller of $(12 - 1)$ and $(15 - 1)$, which is 11. With df = 11 and a two-tailed test, Table A-3 indicates that the *P*-value is greater than 0.20. Technology will provide the *P*-value of 0.5172 when using the original data or unrounded sample statistics.

Example: Are Male Professors and Female Professors Rated Differently by Students? (8 of 10)

Solution

Step 7: Because the *P*-value is greater than the significance level of 0.05, we fail to reject the null hypothesis. ("If the *P* is **low,** the null must go.")

Example: Are Male Professors and Female Professors Rated Differently by Students? (9 of 10)

Interpretation

Step 8: There is not sufficient evidence to warrant rejection of the claim that female professors and male professors have the same mean course evaluation score.

Example: Are Male Professors and Female Professors Rated Differently by Students? (10 of 10)

Technology

The tricky part about the preceding *P*-value approach is that Table A-3 can give only a range for the *P*-value, and determining that range is often somewhat difficult. Technology automatically provides the *P*-value, so technology makes the *P*-value method quite easy. See the accompanying XLSTAT display showing the test statistic of $t = -0.660$ (rounded) and the *P*-value of 0.5172.

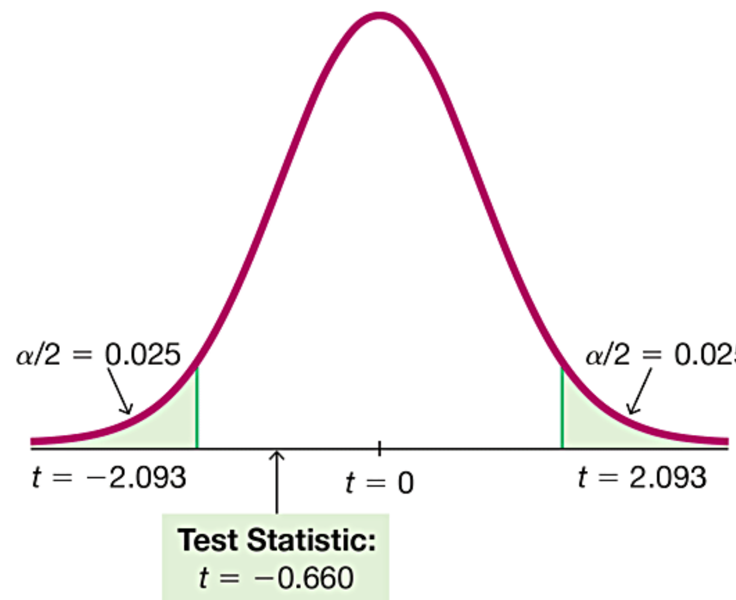
XLSTAT

Difference	-0.1267
t (Observed value)	-0.6599
|t| (Critical value)	2.0926
DF	19
p-value (Two-tailed)	0.5172
alpha	0.05

Critical Value Method

The critical value method of testing a claim about two means is generally easier than the *P*-value method. With $n_1 = 12$ and $n_2 = 15$, the number of degrees of freedom is 11. In Table A-3 with df = 11 and $\alpha = 0.05$ in two tails, we get critical values of $t = \pm 2.201$. Technology provides $t = \pm 2.093$.

The test statistic of $t = -0.660$ falls between the critical values, so the test statistic is not in the critical region and we fail to reject the null hypothesis.



Example: Confidence Interval for Female and Male Course Evaluation Scores (1 of 5)

Using the previous data, construct a 95% confidence interval estimate of the difference between the mean course evaluation score for female professors and the mean course evaluation score for male professors.

Example: Confidence Interval for Female and Male Course Evaluation Scores (2 of 5)

Solution

Requirement check

Because we are using the same data from the previous example, the same requirement check applies here, so the requirements are satisfied.

Example: Confidence Interval for Female and Male Course Evaluation Scores (3 of 5)

Solution

We first find the value of the margin of error *E*. In Table A-3 with df = 11 and $\alpha = 0.05$ in two tails, we get critical values of $t = \pm 2.201$. (Technology can be used to find more accurate critical values of $t = \pm 2.093$.

$$E = t_{\frac{\alpha}{2}} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

$$= 2.201 \sqrt{\frac{0.563001^2}{12} + \frac{0.395450^2}{15}} = 0.422452$$

Example: Confidence Interval for Female and Male Course Evaluation Scores (4 of 5)

Solution

Using $E = 0.422452$, $\bar{x}_1 = 3.866667$, and $\bar{x}_2 = 3.993333$, we can now find the confidence interval as follows:

$$(\bar{x}_1 - \bar{x}_2) - E < (\mu_1 - \mu_2) < (\bar{x}_1 - \bar{x}_2) + E$$

$$-0.55 < (\mu_1 - \mu_2) < 0.30$$

If we use technology to obtain more accurate results, we get the confidence interval of $-0.53 < (\mu_1 - \mu_2) < 0.27$, so we can see that the confidence interval above is quite good.

Example: Confidence Interval for Female and Male Course Evaluation Scores (5 of 5)

Interpretation

We are 95% confident that the limits of −0.53 and 0.27 actually do contain the difference between the two population means. Because those limits contain 0, this confidence interval suggests that there is not a significant difference between the mean course evaluation score for female professors and the mean course evaluation score for male professors.

Alternate Methods: Assume that $\sigma_1 = \sigma_2$ and Pool the Sample Variances (1 of 2)

The **pooled estimate of σ^2** is denoted by s_p^2 and is a weighted average of s_1^2 and s_2^2, which is used in the test statistic for this case:

Test Statistic
$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\dfrac{s_p^2}{n_1} + \dfrac{s_p^2}{n_2}}}$$

where $$s_p^2 = \frac{(n_1 - 1)\, s_1^2 + (n_2 - 1)\, s_2^2}{(n_1 - 1) + (n_2 - 1)}$$ (pooled sample variance)

Degrees of freedom : df = $n_1 + n_2 - 2$.

Alternate Methods: Assume that $\sigma_1 = \sigma_2$ and Pool the Sample Variances (2 of 2)

Confidence interval are found by evaluating

$$(\bar{x}_1 - \bar{x}_2) - E < (\mu_1 - \mu_2) < (\bar{x}_1 - \bar{x}_2) + E$$

Margin of Error for Confidence Interval $E = t_{\frac{\alpha}{2}} \sqrt{\dfrac{s_p^2}{n_1} + \dfrac{s_p^2}{n_2}}$

where $s_p{}^2 = \dfrac{(n_1 - 1)\, s_1{}^2 + (n_2 - 1)\, s_2{}^2}{(n_1 - 1) + (n_2 - 1)}$

Degrees of freedom : df = $n_1 + n_2 - 2$.

Alternate Methods: Assume When σ_1 and σ_2 Are Known (1 of 2)

In reality, the population standard deviations s_1 and s_2 are almost never known, but if they are somehow known, the test statistic and confidence interval are based on the normal distribution instead of the *t* distribution.

Test Statistic $$z = \frac{(\bar{x}_1 \quad \bar{x}_2) \quad (\mu_1 \quad \mu_2)}{\sqrt{\frac{{}^{2}_{1}}{n_1} + \frac{{}^{2}_{2}}{n_2}}}$$

Alternate Methods: Assume When σ_1 and σ_2 Are Known (2 of 2)

Confidence interval are found by evaluating

$$(\bar{x}_1 - \bar{x}_2) - E < (\mu_1 - \mu_2) < (\bar{x}_1 - \bar{x}_2) + E$$

Margin of Error for Confidence Interval

$$E = z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Recommended Strategy for Two Independent Means

Here is the recommended strategy for the methods of this section:

Assume that s_1 and s_2 are unknown, do not assume that $s_1 = s_2$, and use the test statistic and confidence interval given in Part 1 of this section.

Pearson